Estimation of TAR model

①

$$y_t = \phi_1 y_{t-1} I(y_{t-1} > r) + \phi_2 y_{t-1} I(y_{t-1} \le r) + \varepsilon_t$$

$$L(\phi_1, \phi_2, r) = \sum_{t=1}^{n} \left[y_t - \phi_1 y_{t-1} I(y_{t-1} > r) - \phi_2 y_{t-1} I(y_{t-1} \le r) \right]^2$$

$$= \sum_{t=1}^{n} (y_t - \phi_1 y_{t-1}) I(y_{t-1} > r)$$

$$+ \sum_{t=1}^{n} (y_t - \phi_2 y_{t-1}) I(y_{t-1} \le r)$$

$$\hat{\phi}_1(r) = \sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} > r) \Big/ \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > r)$$

$$\hat{\phi}_2(r) = \sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} \le r) \Big/ \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \le r)$$

$-\infty = X_{(0)}$ $X_{(1)}$ $X_{(2)}$ $X_{(n)}$ $+\infty = X_{(n+1)}$

$$\min_{\phi_1, \phi_2} \min_{r} L(\phi_1, \phi_2, r)$$

$$= \min_{\phi_1, \phi_2} \min_{0 \le i \le n} \min_{X_{(i)} \le r < X_{(i+1)}} L(\phi_1, \phi_2, r)$$

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On $r \in [X_{(i)}, X_{(i+1)})$,

$$I(y_{t-1} > r) = \begin{cases} 1 & \text{if } y_t = X_{(i+1)}, \dots \text{ or } X_{(n)} \\ 0 & \text{if } y_t = X_{(1)}, \dots \text{ or } X \end{cases} = I(y_{t-1} > y_{(i)})$$

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$$I(y_{t-1} \le r) = \begin{cases} 1 & \text{if } y_{t-1} = y_{(1)}, \ldots, y_{(i)} \\ 0 & \text{if } y_{t-1} = y_{(i+1)}, \ldots, y_{(n)} \end{cases}$$

$$= I\{y_{t-1} \le y_{(i)}\}$$

$$1 = \min_{0 \le i \le r} \min_{\phi_1, \phi_2} L(\phi_1, \phi_2, y_{(i)})$$

$$\hat{\phi}_{1i} = \sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} > y_{(i)}) \Big/ \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > y_{(i)})$$

$$\hat{\phi}_{2i} = \sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} \le y_{(i)}) \Big/ \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \le y_{(i)})$$

$$1 = \min_{0 \le i \le r} L(\hat{\phi}_{1i}, \hat{\phi}_{2i}, y_{(i)})$$

$$\hat{r} = y_{(k)}, \text{ when } k = \arg\min_{0 \le i \le r} L(\hat{\phi}_{1i}, \hat{\phi}_{2i}, y_{(i)})$$

$③$

$$L(\hat{\phi}_{1n}, \hat{\phi}_{2n}, y_{(n)})$$

$$= \sum_{t=1}^{n} (y_t - \hat{\phi}_{1n} y_{t-1})^2 I(y_{t-1} \geq y_{(n)}) + \sum_{t=1}^{n} (y_t - \hat{\phi}_{2n} y_{t-1})^2 I(y_{t-1} \leq y_{(n)})$$

$$\sum_{t=1}^{n} (y_t - \hat{\phi}_{2n} y_{t-1})^2 I(y_{t-1} \leq y_{(n)})$$

$$= \sum_{t=1}^{r} \left[\varepsilon_t - (\hat{\phi}_{2n} - \phi_2) y_{t-1} \right]^2 I(y_{t-1} \leq y_{(n)}) I(y_{t-1} \leq r_0)$$

$$+ \sum_{t=1}^{r} \left[\varepsilon_t - (\hat{\phi}_{2n} - \phi_1) y_{t-1} \right]^2 I(y_{t-1} \leq y_{(n)}) I(y_{t-1} > r_0)$$

$$\hat{\phi}_{2n} - \phi_2 = \frac{\sum_{t=1}^{r} (y_t - \phi_2 y_{t-1}) y_{t-1} I(y_{t-1} \leq y_{(n)})}{\sum_{t=1}^{r} y_{t-1}^2 I(y_{t-1} \leq y_{(n)})}$$

$$= \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} \leq y_{(n)})$$

$$- \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \leq y_{(n)}) I(y_{t-1} \leq r) (\phi_2 - \phi_{10})$$

$$- \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \leq y_{(n)}) I(y_{t-1} \geq r) (\phi_2 - \phi_{10})$$

$$\hat{\phi}_{1n} - \phi_1 = ?$$

(4)

$$\varepsilon_t(\theta) = y_t - \phi_1 y_{t-1} I(y_{t-1} > r) - \phi_2 y_{t-1} I(y_{t-1} \le r)$$

$$= \varepsilon_t + \phi_{10} y_{t-1} I(y_{t-1} > r) + \phi_{20} y_{t-1} I(y_{t-1} \le r)$$

$$- \phi_1 y_{t-1} I(y_{t-1} > r_0) \mp \phi_2 y_{t-1} I(y_{t-1} \le r_0)$$

$$\equiv \varepsilon_t - X_t(\theta)$$

$$X_t(\theta) = \phi_1 y_{t-1} I(y_{t-1} > r) + \phi_2 y_{t-1} I(y_{t-1} \le r)$$

$$- \phi_{10} y_{t-1} I(y_{t-1} > r_0) - \phi_{20} y_{t-1} I(y_{t-1} \le r_0)$$

$$E\varepsilon_t^2(\theta) = E\varepsilon_t^2(\theta_0) - E X_t^2(\theta) \ge E\varepsilon_t^2(\theta_0) = \sigma^2$$

"=" holds

$$\iff X_t(\theta) = 0 \qquad (r_0 < r)$$

$$X_t(\theta) = (\phi_1 - \phi_{10}) y_{t-1} I(y_{t-1} > r) + \phi_{10} y_{t-1} [I(y_{t-1} > r) - I(y_{t-1} > r_0)]$$

$$- I(r_0 < y_{t-1} \le r)$$

$$+ (\phi_2 - \phi_{20}) y_{t-1} I(y_{t-1} \le r) + \phi_{20} y_{t-1} [I(y_{t-1} \le r) - I(y_{t-1} \le r_0)]$$

$$I(r_0 < y_{t-1} \le r)$$

$$= (\phi_1 - \phi_{10}) y_{t-1} I(y_{t-1} > r)$$

$$+ (\phi_{20} - \phi_{10}) y_{t-1} I(r_0 < y_{t-1} \le r)$$

$$+ (\phi_2 - \phi_{20}) y_{t-1} I(r_0 < y_{t-1} \le r) + (\phi_2 - \phi_{20}) y_{t-1} I(y_{t-1} \le r_0)$$

$$= 0$$

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$\Longleftrightarrow \quad (\phi_1 - \phi_{10}) y_{t-1} I(y_{t-1} > r) = 0$ ①

$[\phi_{20} - \phi_{10} + \phi_2 - \phi_{20}] y_{t-1} I(r_0 < y_{t-1} \le r) = 0$ ②

$(\phi_2 - \phi_{20}) y_{t-1} I(y_{t-1} \le r_0) = 0$ ③

~~If $I(y_{t-1} > r) = 1$, then $\phi_1 = \phi_{10}$~~

By (1)

$$(\phi_1 - \phi_{10}) E\, y_{t-1} I(y_{t-1} > r) = 0$$

But

$$E\, y_{t-1} I(y_{t-1} > r)$$

$$= E[(\varepsilon_t + g_t) I(\varepsilon_t > r - g_t)]$$

$$= E \int_{r-g_t}^{\infty} (x + g_t) f(x)\, dx \qquad (z = x + g_t$$

$$= E \int_{r}^{\infty} z f(z - g_t)\, dz > 0 \quad \text{since } f(x) > 0 \text{ a.s.}$$

$\Longrightarrow \quad (\phi_1 - \phi_{10}) = 0.$

Similarly $\phi_2 - \phi_{20} = 0$. by (3)

By (2)

$$(\phi_2 - \phi_{10})\, y_{t-1} I(r_0 < y_{t-1} \le r) = 0$$

Since $\phi_2 = \phi_{20}$ and $\phi_{20} \neq \phi_{10}$, we have

$$y_{t-1} I(r_0 < y_{t-1} \leq r) = 0$$

$$E\, y_{t-1} I(r_0 < y_{t-1} \leq r)$$

$$= E \int_{r_0}^{r} z f(z - g_t)\, dz = 0$$

$$\Rightarrow \quad r = r_0$$

Thus, $E\, \varepsilon_t^2(\theta)$ achieve its minimum value at $\theta = \theta_0 = (\phi_{10}, \phi_{20}, r_0)$.

Assume that the parametric space Θ is a compact set and θ_0 is an interior point.

For any neighborhood V of $\theta_0 \subseteq \Theta$,

$$\inf_{\theta \in V^c} E\, \varepsilon_t^2(\theta) \equiv C > E\, \varepsilon_t^2(\theta_0)$$

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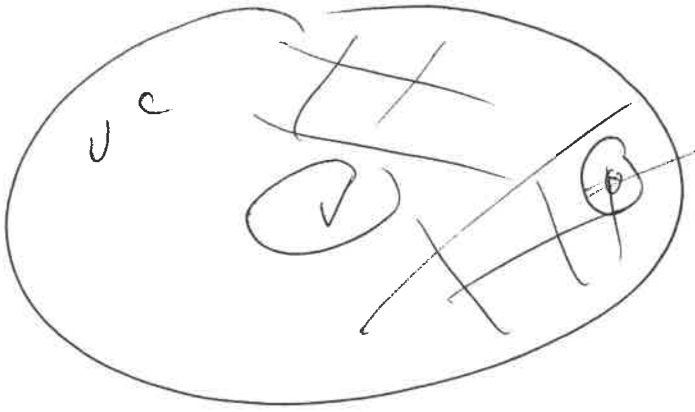

$$E \sup_{\theta^* \in \mathcal{U}_\theta(\eta)} \left| \varepsilon_t^2(\theta^*) - \varepsilon_t^2(\theta) \right| \to 0 \quad \text{as } \eta \to 0$$

φ

$$\varepsilon_t(\theta^*) - \varepsilon_t(\theta) = X_t(\theta) - X_t(\theta^*)$$

$r^* > r$

$r \qquad r^*$

$$= \phi_1^* y_{t-1} (y_{t-1} > r^*) + \phi_2^* y_{t-1} I(y_{t-1} \le r^*)$$
$$- \phi_1 y_{t-1} I(y_{t-1} > r) - \phi_2 y_{t-1} I(y_{t-1} \le r)$$

$$= (\phi_1^* - \phi_1) y_{t-1} I(y_{t-1} > r^*)$$
$$+ (\phi_2 - \phi_1) y_{t-1} I(r < y_{t-1} \le r^*)$$
$$+ (\phi_2^* - \phi_2) y_{t-1} I(r < y_{t-1} \le r^*) + (\phi_2^* - \phi_2) y_{t-1} I(y_{t-1} \le r)$$
$$= (\phi_1^* - \phi_1) y_{t-1} I(y_{t-1} > r^*)$$
$$+ (\phi_2^* - \phi_1) y_{t-1} I(r < y_{t-1} \le r^*)$$
$$+ (\phi_2^* - \phi_2) y_{t-1} I(y_{t-1} \le r)$$

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$$E \sup_{\theta^\alpha \in U_\theta(\eta)} \left| \Sigma_t^2(\theta^\alpha) - \Sigma_t(\theta) \right|^2$$

$$\le 2 \widehat{\sup} (\phi_1^\alpha - \phi_1)^2 E\, y_{t-1}^2 + 2 \sup (\phi_2^\alpha - \phi_2)^2 E\, y_{t-1}^2$$

$$+ 2(\phi_1^2 + \eta^2)\, E\, y_{t-1}^2 I(r < y_{t-1} \le r^\alpha)$$

$$\to 0 \quad \text{as} \quad \eta \to 0.$$

So, (4) holds.

$$\bigcup_{j=1}^{N} U_\theta(\eta_0) \text{ covers } V^c.$$

For $\forall \varepsilon > 0$, $\exists n_0$, such that, as $n \ge n_0$,

$$\inf_{\theta^\alpha \in U_\theta(\eta_0)} \left[\frac{1}{n} \sum_{t=1}^{n} \Sigma_t^2(\theta^\alpha) \right]$$

$$\ge \frac{1}{n} \sum_{t=1}^{n} \inf_{\theta^\alpha \in U_\theta(\eta)} \Sigma_t^2(\theta^\alpha)$$

$$\ge E \inf_{\theta^\alpha \in U_\theta(\eta)} \Sigma_t^2(\theta^\alpha) - \varepsilon$$

$$\ge E\, \Sigma_t^2(\theta) - E \sup_{\theta^\alpha \in U_\theta(\eta)} \left| \Sigma_t^2(\theta^\alpha) - E\,\Sigma_t^2(\theta) \right| - \varepsilon$$

$$\ge E\, \Sigma_t^2(\theta) - 2\varepsilon$$

$$= E\,\varepsilon_t^2(\theta) - 2\xi + \left[C - E\,\varepsilon_t^2(\theta_0)\right]$$

$$\geq \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^2(\theta_0) \underbrace{-3\xi + \left[C - E\,\varepsilon_t^2\right]}_{C^\alpha > 0}$$

$$\inf_{\theta\in V^c}\left[\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2(\theta)\right]$$

$$\geq \min_{1\leq j\leq T}\ \inf_{\theta^\alpha\in U_\theta(\eta_0)}\left[\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2(\theta)\right]$$

$$\geq \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2(\theta_0) + C^\alpha > \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2(\theta)$$

This means that, as $n > n_0$,

$$\hat{\theta}_n \in U \quad a.s$$

$$\therefore \quad \hat{\theta}_n \longrightarrow \theta_0 \quad a.s. \qquad \square$$