$$\varepsilon_t(\theta) = \varepsilon_t(\lambda, r), \text{ where } \lambda = (\phi_1, \phi_2)$$

$$L_n(\lambda, r) = \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^2(\lambda, r)$$

$$G(u) = E\, I(r_0 < y_t \le r_0 + u)$$

Given any $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$P(|n(\hat{r}_n - r_0)| \ge B)$$

$$\le P(|n(\hat{r}_n - r_0)| \ge B,\ \|\hat{\lambda}_n - \lambda_0\| \le \delta) + \frac{\varepsilon}{2}.$$

$$P(n|\hat{r}_n - r_0| \ge B,\ \|\hat{\lambda}_n - \lambda_0\| \le \delta)$$

$$= P(n|\hat{r}_n - r_0| \ge B;\ \|\hat{\lambda}_n - \lambda_0\| \le \delta\ \ G(|\hat{r}_n - r_0|) \ge G(\tfrac{B}{n}))$$

$$= P\left(\begin{array}{c} n|\hat{r}_n - r_0| \ge B,\ \|\hat{\lambda}_n - \lambda_0\| \le \delta,\ G(|\hat{r}_n - r_0|) \ge G(\tfrac{B}{n}), \\ L_n(\hat{\lambda}_n, \hat{r}_n) - L_n(\hat{\lambda}_n, r_0) \le 0 \end{array}\right)$$

$$\le P\left[\inf_{\substack{n|r - r_0| \ge B \\ \|\lambda - \lambda_0\| \le \delta}} \frac{L_n(\lambda, r) - L_n(\lambda, r_0)}{G(|r - r_0|)} \le 0\right]$$

~~$$P\left[\inf_{\substack{n|r - r_0| \ge B \\ \|\lambda - \lambda_0\| \le \delta}} \frac{L_n(\lambda, r) - L_n(\lambda, r_0)}{G(|r - r_0|)} \le \gamma\right] < \varepsilon$$~~

~~for $\forall\, \gamma > 0$.~~

$$\varepsilon_t(\lambda, r) - \varepsilon_t(\lambda, r_0) \qquad [r > r_0]$$

$$= y_t - \phi_1 y_{t-1} I(y_{t-1} > r) - \phi_2 y_{t-1} I(y_{t-1} > r)$$

$$- [y_t - \phi_1 y_{t-1} I(y_{t-1} > r_0) - \phi_2 y_{t-1} I(y_{t-1} > r_0)$$

$$= -\phi_1 y_{t-1} I(r_0 < y_{t-1} \leq r) + \phi_2 y_{t-1} (r_0 < y_{t-1} \leq r)$$

$$= (\phi_2 - \phi_1) y_{t-1} I(r_0 < y_{t-1} \leq r)$$

$$L_n(\lambda, r) - L_n(\lambda, r_0)$$

$$= \frac{1}{n} \sum_{t=1}^{n} [\varepsilon_t^2(\lambda, r) - \varepsilon_t^2(\lambda, r_0)]$$

$$= \frac{1}{n} \sum_{t=1}^{n} \Big[[\varepsilon_t(\lambda, r) - \varepsilon_t(\lambda, r_0)]^2 + 2[\varepsilon_t(\lambda, r) - \varepsilon_t(\lambda, r_0)] \varepsilon_t(\lambda, r_0) \Big]$$

$$= \frac{1}{n} \sum_{t=1}^{n} (\phi_2 - \phi_1)^2 y_{t-1}^2 I(r_0 < y_{t-1} \leq r)$$

$$+ \frac{2}{n} \sum_{t=1}^{n} (\phi_2 - \phi_1) y_{t-1} I(r_0 < y_{t-1} \leq r) \varepsilon_t(\lambda, r_0)$$

$$I(r_0 < y_{t-1} \leq r) \varepsilon_t(\lambda, r_0)$$

$$= I(r_0 < y_{t-1} \leq r) \Big[\varepsilon_t + \phi_{10} y_{t-1} I(y_{t-1} > r_0) + \phi_{20} y_{t-1} I(y_{t-1} \leq r_0) - \phi_1 y_{t-1} I(y_{t-1} > r_0) - \phi_2 y_{t-1} I(y_{t-1} \leq r_0) \Big]$$

$$= I(r_0 < y_{t-1} \leq r) \varepsilon_t + (\phi_{20} - \phi_2) y_{t-1} I(r_0 \leq y_{t-1} \leq r)$$

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$$L_n(\lambda, r) - L_n(\lambda, r_0)$$

$$= \frac{1}{n}\sum_{t=1}^{n}(\phi_2-\phi_1)^2 y_{t-1}^2 I(r_0 < y_{t-1} \le r)$$

$$+ \frac{2(\phi_2-\phi_1)}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(r_0 < y_{t-1} \le r)$$

$$+ \frac{2(\phi_2-\phi_1)(\phi_{20}-\phi_2)}{n}\sum_{t=1}^{n} y_{t-1}^2 I(r_0 \le y_{t-1} \le r)$$

$$\cancel{= (\phi_2-\phi_1)^2 G_{1n}} +$$

$$= \left[(\phi_2-\phi_1)^2 + 2(\phi_2-\phi_1)(\phi_{20}-\phi_2)\right] G_{1n}^{*}(r)$$

$$+ 2(\phi_2-\phi_1)\, G_{2n}(r),$$

where $G_{1n}^{*}(r) = \frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 I(r_0 \le y_{t-1} \le r)$

$$G_{2n}(r) = \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(r_0 \le y_{t-1} \le r)$$

On $\{\lambda : \|\lambda-\lambda_0\| \le \delta\}$,

$$(\phi_2-\phi_1)^2 + 2(\phi_2-\phi_1)(\phi_{20}-\phi_2) \longrightarrow (\phi_{20}-\phi_{10})^2 \equiv c > 0$$

$$|\phi_2-\phi_1| \longrightarrow |\phi_{20}-\phi_{10}| = \sqrt{c}.$$

Thus when δ is small enough,

$r_0 \neq 0$ (13)

$\frac{\sqrt{n}(\hat{\beta}_r - 0)}{G(u)} \to ?$

$\cancel{= \sum \psi_t^2 I(r_0 < y_{t-1} \le r_0 + u)}$

$L_n(\lambda, r) - L_n(\lambda, r_0)$

$$\ge \frac{1}{2} c\, G_{1n}^{2}(r) - 2\sqrt{c}\, |G_{2n}^{(r)}|$$

$$\ge \frac{1}{2} c r_0^2 \left[G_{1n}^{(r)}\right] - 2\sqrt{c}\, |G_{2n}^{(r)}|, \quad (\text{when } r_0 \neq 0)$$

where $G_{1n}^{(r)} = \frac{1}{n} \sum_{t=1}^{n} I(0 < y_{t-1} \le r)$.

Thus, we can take δ small enough,

$$\inf_{\substack{n|r-r_0| \ge B \\ \|\lambda - \lambda_0\| \le \delta}} \frac{L_n(\lambda, r) - L_n(\lambda, r_0)}{G(|r-r_0|)}$$

$$\ge \frac{c r_0^2}{2} \inf_{n|r-r_0| > B} \frac{G_{1n}(r)}{G(|r-r_0|)} - \sqrt{c} \sup_{n|r-r_0| \ge B} \left|\frac{G_{2n}(r)}{G(|r-r_0|)}\right|$$

$$\ge \frac{c r_0^2}{2} - \frac{c r_0^2}{2} \max_{n|r-r_0| > B} \left|\frac{G_{1n}(r)}{G(|r-r_0|)} - 1\right|$$

$$- \sqrt{c} \sup_{n|r-r_0| \ge B} \left|\frac{G_{2n}(r)}{G(|r-r_0|)}\right|$$

We only need to prove that

for $\forall \varepsilon > 0, \eta > 0, \exists B$ s.t.

$$P\left(\sup_{n|r-r_0| \geq B} \left| \frac{G_{1n}(r) - G(|r-r_0|)}{G(|r-r_0|)} \right| > \varepsilon \right) < \eta \quad (1)$$

and

$$P\left(\sup_{n|r-r_0| \geq B} \left| \frac{G_{2n}(r)}{G(|r-r_0|)} \right| > \varepsilon \right) < \eta \quad (2)$$

as $n \to \infty$.

To be simple, Assume that $r_0 = 0$ and $G(x) = x$ and consider the case with $r > 0$.

$$(1) = P\left(\sup_{\frac{B}{n} \leq r \leq \delta} \left| \frac{1}{n G(r)} \sum_{t=1}^{n} I(0 < y_{t-1} \leq r) - 1 \right| > \varepsilon \right)$$

$$\overset{y = \frac{nr}{B}}{=} P\left(\sup_{1 \leq y \leq \frac{n\delta}{B}} \left| \frac{1}{n G(\frac{yB}{n})} \sum_{t=1}^{n} I\left(0 < y_{t-1} \leq \frac{yB}{n}\right) - 1 \right| > \varepsilon \right)$$

$$\leq P\left(\sup_{i \in \mathbb{N}} \sup_{b^i \leq y \leq b^{i+1}} \left| \frac{1}{n G(\frac{yB}{n})} \sum_{t=1}^{n} I\left(0 < y_{t-1} \leq \frac{yB}{n}\right) - 1 \right| > \varepsilon \right)$$

$$\le \sum_{i=0}^{\infty} P\left(\sup_{b^i \le y \le b^{i+1}} \left| \frac{1}{n G(\frac{y\beta}{n})} \sum_{t=1}^{n} I\left(0 < Y_{t-1} \le \frac{y\beta}{n}\right) - 1 \right| > \varepsilon \right) \quad (3)$$

For $\forall\, 0 < x < y < xb$, with

$$\left| \frac{G_{1n}(x)}{G(x)} - 1 \right| < \eta_0 \quad \text{and} \quad \left| \frac{G_n(bx)}{G(bx)} - 1 \right| < \eta_0.$$

$$\frac{G_{1n}(y)}{G(y)} - 1 = \frac{b\, G_n(y)}{by} - 1 \le b \left[\frac{G_n(bx)}{bx} \right] - 1$$

$$\le b(\eta_0 + 1) - 1$$

$$\frac{G_{1n}(y)}{G(y)} - 1 \ge \frac{G_n(x)}{bx} - 1 \ge (1-\eta_0) b^{-1} - 1.$$

Take $b > 1$ and a small η_0 s.t.

$$\min\{ 1 - (1-\eta_0) b^{-1},\ b(\eta_0 + 1) - 1 \} < \eta.$$

~~When $0 < x < y < bx$,~~

Thus, $$\sup_{x < y \le bx} \left| \frac{G_{1n}(y)}{G(y)} - 1 \right| < \eta.$$

Thus, when $0 < x < y < bx$,

$$\left\{ \left| \frac{G_{1n}(y)}{G(y)} - 1 \right| \geq \eta \right\}$$

$$\subseteq \left\{ \left| \frac{G_{1n}(x)}{G(x)} - 1 \right| \geq \eta_0 \right\} \vee \left\{ \left| \frac{G_{1n}(bx)}{G(bx)} - 1 \right| \geq \eta_0 \right\}$$

$$P\left(\sup_{b^i < y \leq b^{i+1}} \left| \frac{G_{1n}\left(\frac{yB}{n}\right)}{G\left(\frac{yB}{n}\right)} - 1 \right| > \varepsilon \right)$$

$$\leq P\left(\left| \frac{G_{1n}\left(\frac{b^iB}{n}\right)}{G\left(\frac{b^iB}{n}\right)} - 1 \right| > \varepsilon \right)$$

$$+ P\left(\left| \frac{G_{1n}\left(\frac{b^{i+1}B}{n}\right)}{G\left(\frac{b^{i+1}B}{n}\right)} - 1 \right| > \varepsilon \right)$$

Thus ① + ③,

$$① \leq P\left(\left| \frac{G_{1n}\left(\frac{B}{n}\right)}{G\left(\frac{B}{n}\right)} - 1 \right| > \varepsilon \right)$$

$$+ 2 \sum_{i=1}^{n} P\left(\left| \frac{G_n\left(\frac{b^iB}{n}\right)}{G\left(\frac{b^iB}{n}\right)} - 1 \right| > \varepsilon \right).$$

$$P\left(\left|\frac{G_n(x)}{G(x)}-1\right|>\varepsilon\right)$$

$$=P\left(\left|G_n(x)-G(x)\right|>\varepsilon\, G(x)\right)$$

$$\le \frac{1}{\varepsilon^2 G^2(x)}\, E\left|G_n(x)-G(x)\right|^2 \qquad (\times\times)$$

$$E\left|G_n(x)-G(x)\right|^2=E\left\{\frac{1}{n}\sum_{t=1}^{n}\left[I(0<y_{t_i-1}\le x)-G(x)\right]\right\}^2$$

$$=\frac{1}{n^2}\sum_{t=1}^{n}E\left[I(0<y_{t_i-1}<x)-G(x)\right]^2$$

$$+\frac{1}{n^2}\sum_{i\ne j}E\left[I(0<y_{t_i-1}<x)-G(x)\right]\left[I(0<y_{t_j-1}<x)-G(x)\right]$$

$$=U_{1n}+U_{2n}$$

$$U_{1n}=\frac{1}{n}E\left[I(0<y_{t_i-1}<x)-G(x)\right]^2$$

$$=\frac{1}{n}\left\{E\, I(0<y_{t_i-1}<x)-G^2(x)\right. < \frac{1}{n}G(x)$$

$$U_{2n}=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{r=1}^{n-i}\left[E\, I(0<y_{t_i-1}<x)\, I(0<y_{t_{i-1-r}}<x)-G^2(x)\right]$$

$$E\, I(0<y_{t_i-1}<x)\, I(0<y_{t_{i-r}}<x)$$

$$=E\left\{E\left[I(0<y_{t_i-1}<x)\,\middle|\,\mathcal{F}_{t_{i-r+1}}\right] I(0<y_{t_{i-r}-1}<x)\right]\right\}$$

$$E\left[I(0 < y_{t-i-1} < x) \mid \mathcal{F}_{t-i-r+1} \right]$$

$$= P^r(y_{t-i-r+1},\ 0 < y_{t-i-1} < x \mid \mathcal{F}_{t-i-r+1})$$

$$\leq G(x) + \left\| P^r(y_{t-i-r+1},\ 0 < y_{t-i-1} < x \mid \mathcal{F}_{t-i-r+1}) - G(x) \right\|$$

$$\leq G(x) + K(1 + |y_{t-i-r+1}|)\rho^r$$

$$E\, I(0 < y_{t-i-1} < x)\, I(0 < y_{t-i-r} < x)$$

$$\leq G^2(x) + K\, E(1 + |y_{t-i-r-1}|)\, I(0 < y_{t-i-r-1} < x)\, \rho^r$$

$$\leq G^2(x) + H^* \rho^r G(x)$$

Thus. $U_{2n} \leq H^* G(x) \cdot \frac{1}{n^2} \sum_{i=1}^{n} \sum_{r=1}^{n-i} \rho^r = \frac{H^* G(x)}{n}$

(**) $E\left| G_n(x) - G(x) \right|^2 \leq \frac{H}{n} G(x).$

Thus. $P\left(\left| \frac{G_n(x)}{G(x)} - 1 \right| > \varepsilon \right) \leq \frac{H}{n \varepsilon^2 G(x)}$

$$\| p^n(x, \cdot) - \pi(\cdot) \| \leq K(1 + |x|)\rho^n$$

K is indepe of x

(19)

Thus, $P\left(\left| \frac{G_n(\frac{B}{n})}{G(\frac{B}{n})} - 1 \right| > \varepsilon \right) \leq \frac{H}{n\varepsilon^2 \cdot \frac{B}{n}} = \frac{H}{\varepsilon^2 B}$

$(G(x) = x)$

~~as B is large enough.~~

$$P\left(\left| \frac{G_n(\frac{b^i B}{n})}{G(\frac{b^i B}{n})} - 1 \right| > \varepsilon \right) \leq \frac{H}{n\varepsilon^2 \cdot \frac{b^i B}{n}} = \frac{H}{\varepsilon^2 B b^i}$$

(*)

$$P\left(\sup_{s > r \geq \frac{B}{n}} \left| \frac{\frac{1}{n}\sum_{t=1}^{n} I(0 < y_{t,1} < r)}{G(r)} - 1 \right| > \varepsilon \right)$$

$$\leq \frac{H}{\varepsilon^2 B} + \frac{2H}{\varepsilon^2 B} \sum_{i=1}^{\infty} \frac{1}{b^i} < \varepsilon_0$$ as B is large enough.

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$$\hat{\phi}_1 = \frac{\sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} > \hat{r})}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r})} \qquad \hat{r} > r_0$$

$$\hat{\phi}_2 = \frac{\sum_{t=1}^{n} y_t y_{t-1} I(y_{t-1} \leq \hat{r})}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \leq \hat{r})}$$

$$\cancel{\sum_{t=1}^{n} y_t y} \qquad \hat{\phi}_1 - \phi_{10} = \frac{\sum_{t=1}^{n} (y_t - \phi_{10} y_{t-1}) y_{t-1} I(y_{t-1} > \hat{r})}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r})}$$

$$= \frac{\sum_{t=1}^{n} (\varepsilon_t + \cancel{\phi_{10} y_{t-1} - \phi_{10} y_{t-1}}) y_{t-1} I(y_{t-1} > \hat{r})}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r})}$$

$$\sqrt{n}(\hat{\phi}_1 - \phi_{10}) = \frac{\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > \hat{r})}{\frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r})}$$

$$\frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r}) = \frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > r_0) - \frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} < \hat{r})$$

$$= E\, y_{t-1}^2 I(y_{t-1} > r_0) + o_p(1) - \frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} < \hat{r})$$

On the event $\{ r_0 < \hat{r} < r_0 + \frac{B}{n} \}$

$$\frac{1}{n} \sum_{t=1}^{n} y_t^2 I(r_0 < y_{t-1} < \hat{r}) \leq \frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \leq \frac{B}{n})$$

$$P\left[\frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 I\left(r_0 < y_{t-1} \le \frac{B}{n}\right) > \varepsilon\right]$$

$$\le \frac{n\varepsilon}{n} E\, y_t^2 I\left(r_0 < y_{t-1} \le \frac{B}{n}\right) \to 0 \quad <$$

So. $\frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r}) = E\, y_{t-1}^2 I(y_{t-1} > r_0)$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(y_{t-1} > \hat{r})$$

$$= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(y_{t-1} > r_0) - \underline{\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(r_0 < y_{t-1} \le \hat{r})}$$

On the view $\{r_0 < \hat{r} < r_0 + \frac{B}{n}\}$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} |y_{t-1}\varepsilon_t|\, I(r_0 < y_{t-1} \le \hat{r}) \le \frac{1}{\sqrt{n}}\sum_{t=1}^{n} |y_{t-1}\varepsilon_t|\, I\left(r_0 < y_{t-1} \le \frac{B}{n}\right)$$

$$\xrightarrow{a.s} 0.$$

So $\sqrt{n}(\hat{\phi}_1 - \phi_{10}) \xrightarrow{d} N\left(0, \left[E\, y_{t-1}^2 I(y_{t-1} > r_0)\right]^{-1}\sigma^2\right)$

Similarly $\sqrt{n}(\hat{\phi}_2 - \phi_{20}) \xrightarrow{d} N\left(0, \left[E\, y_{t-1}^2 I(y_{t-1} \le r_0)\right]^{-1}\sigma^2\right)$

(2)

$(\hat{r} > r_0)$

$$L(\hat{\phi}_1, \hat{\phi}_2, \hat{r}) = \sum_{t=1}^{n} (y_t - \hat{\phi}_1 y_{t-1})^2 I(y_{t-1} > \hat{r}) [I(y_{t-1} > r_0) + I(y_{t-1} \le r_0)]$$
$$+ \sum_{t=1}^{n} (y_t - \hat{\phi}_2 y_{t-1}^2) I(y_{t-1} \le \hat{r}) [I(y_{t-1} > r_0) + I(y_{t-1} \le r_0)]$$

$$= \sum_{t=1}^{n} [\varepsilon_t - (\hat{\phi}_1 - \phi_{10}) y_{t-1}]^2 I(\cancel{r_0 < y_{t-1}}\, y_{t-1} > \hat{r})$$

$$+ \sum_{t=1}^{n} [\varepsilon_t - (\hat{\phi}_2 - \phi_{10}) y_{t-1}]^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$+ \sum_{t=1}^{n} [\varepsilon_t - (\hat{\phi}_2 - \phi_{20}) y_{t-1}]^2 I(y_{t-1} \le r_0)$$

$$\equiv B_{1n} + B_{2n} + B_{3n}$$

$$B_{1n} = \sum_{t=1}^{n} \varepsilon_t^2 I(y_{t-1} > \hat{r}) - 2(\hat{\phi}_1 - \phi_{10}) \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > \hat{r})$$
$$+ (\hat{\phi}_1 - \phi_{10})^2 \sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > \hat{r})$$

$$= \sum_{t=1}^{n} \varepsilon_t^2 I(y_{t-1} > \hat{r}) - 2\sqrt{n}(\hat{\phi}_1 - \phi_{10}) \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > r_0) + o_p(1)\right]$$
$$+ [\sqrt{n}(\hat{\phi}_1 - \phi_{10})]^2 \left[\frac{1}{n} \sum_{t=1}^{n} y_{t-1}^2 \cancel{\varepsilon_t} I(y_{t-1} > r_0) + o_p(1)\right]$$

$$= \sum_{t=1}^{n} \varepsilon_t^2 I(y_{t-1} > \hat{r}) - [\sqrt{n}(\hat{\phi}_1 - \phi_{10})]^2 [E\, y_{t-1}^2 I(y_{t-1} > r_0) + o_p(1)]$$

$$B_{2n} = \sum_{t=1}^{n} \varepsilon_t^2 I(y_{t-1} \le r_0) - [\sqrt{n}(\hat{\phi}_2 - \phi_{20})]^2 [E\, y_{t-1}^2 I(y_{t-1} \le r_0) + o_p(1)]$$

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$$\therefore R_{2n} = \sum_{t=1}^{n} \varepsilon_t^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$- 2 \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r_0 < y_{t-1} \le \hat{r}) \cdot (\hat{\phi}_2 - \phi_{0})$$

$$+ \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} < \hat{r}) \cdot (\hat{\phi}_2 - \phi_{10})^2$$

$$= C_{1n} + C_{2n} + C_{3n}$$

$$C_{2n} = -2(\phi_{20} - \phi_{10}) \cdot \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r_0 < y_{t-1} < \hat{r}) \cdot \cancel{\frac{(\hat{\phi}_2 - \phi_{20})}{\sqrt{n}}}$$

$$- 2\sqrt{n}(\hat{\phi}_2 - \phi_{20}) \cdot \frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r_0 < y_{t-1} < \hat{r})$$

$$= -2(\phi_{20} - \phi_{10}) \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r_0 < y_{t-1} < \hat{r}) + o_p(1)$$

$$C_{3n} = (\phi_{20} - \phi_{10})^2 \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$- 2(\phi_{20} - \phi_{10})(\hat{\phi}_2 - \phi_{20}) \cdot \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$+ (\hat{\phi}_{20} - \phi_{20})^2 \cdot \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$= (\phi_{20} - \phi_{10})^2 \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \le \hat{r}) + o_p(1)$$

Thus $$L(\hat{\phi}_1, \hat{\phi}_2, \hat{r}) = \sum_{t=1}^{n} \varepsilon_t^2 - [\sqrt{n}(\hat{\phi}_1 - \phi_{10})]^2 E y_{t-1}^2 I(y_{t-1} > r_0)$$

$$- [\sqrt{n}(\hat{\phi}_2 - \phi_{20})]^2 E y_{t-1}^2 I(y_{t-1} \le r_0)$$

$$+ (\phi_{20} - \phi_{10})^2 \sum_{t=1}^{n} y_{t-1}^2 I(r_0 < y_{t-1} \le \hat{r})$$

$$- 2(\phi_{20} - \phi_{10}) \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r_0 < y_{t-1} \le \hat{r}) + o_p(1)$$

$$= A_n + o_p(1)$$

$$+ \sum_{t=1}^{n} \xi_{1t} I(\gamma_0 < y_{t-1} \leq \hat{\gamma}) + o_p(1)$$

$$\xi_{1t} = (\phi_{20} - \phi_{10})^2 y_{t-1}^2 - 2(\phi_{20} - \phi_{10}) y_{t-1} \varepsilon_t .$$

Similarly, when $\hat{\gamma}_n < \gamma_0$, we can show that

$$L(\hat{\phi}_1, \hat{\phi}_2, \hat{\gamma}) = A_n + o_p(1)$$

$$+ \sum_{t=1}^{n} \xi_{2t} (\hat{\gamma} < y_t \leq \gamma_0)$$

$$\xi_{2t} = (\phi_{10} - \phi_{20})^2 y_{t-1}^2 - 2(\phi_{10} - \phi_{20}) y_{t-1} \varepsilon_t .$$

Denot

$$\hat{\Sigma}(r) = \sum_{t=1}^{n} \xi_{1t} I(r_0 < y_{t-1} \leq \hat{r}) + \sum_{t=1}^{n} \xi_{2t} I(\hat{r} < y_{t-1} \leq r_0)$$

Then, $\hat{r}$ is the minimizer of $\hat{\Sigma}(r)$, i.e.

$$\hat{r} = \arg_r \tilde{L}(r)$$

Let $\hat{u} = n(\hat{r} - r_0)$ and $u = n(r - r_0)$, then $\hat{u}$ is the minimizer of

$$\Sigma^*(u) = \sum_{t=1}^{n} \xi_{1t} I\left(r_0 < y_{t-1} \leq r_0 + \frac{u}{n}\right)$$

$$+ \sum_{t=1}^{n} \xi_{2t} I\left(r_0 + \frac{u}{n} < y_{t-1} \leq r_0\right).$$

i.e.

$$\hat{u} = \arg \min_{u \in [-\Gamma, \Gamma]} L^{+}(u).$$

Given $y_{t-1} = r_0$,

$$\xi_{1t} = (\phi_{20} - \phi_{10})^2 r_0^2 - 2(\phi_{20} - \phi_{10}) r_0 \varepsilon_t \sim F_1(x)$$

$$\xi_{2t} = (\phi_{10} - \phi_{20})^2 r_0^2 - 2(\phi_{10} - \phi_{20}) r_0 \varepsilon_t \sim F_2(x)$$

Define a two-side compound Poisson process

$$\mathcal{P}(z) = I(z<0) \sum_{i=1}^{N_1(-z)} \xi_{1t}^{i} + I(z \geq 0) \sum_{j=1}^{N_2(z)} \xi_{2t}^{j}.$$

where $\{N_1(z) : z \geq 0\}$ and $\{N_2(z) : z \geq 0\}$ are two independent Poisson processes with $N_1(0) = N_2(0) = 0$ and with identical jump size $\pi(r_0)$, and $\pi(x)$ is the density function of y_t, $\{\xi_{1t}^{i}\}$ and $\{\xi_{2t}^{j}\}$ are i.i.d and independent each other. $\xi_{1t}^{i} \sim F_1(x)$, $\xi_{2t}^{j} \sim F_2(x)$.

We can show that

$$L^{+}(u) \Longrightarrow \mathcal{P}(u) \quad \text{on } D[-\Gamma, \Gamma]$$

for any $\Gamma > 0$.