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On functional limits of short- and long-memory linear processes with GARCH(1,1) noises[☆]

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Abstract

This paper considers the short- and long-memory linear processes with GARCH (1,1) noises. The functional limit distributions of the partial sum and the sample autocovariances are derived when the tail index α is in $(0, 2)$, equal to 2, and in $(2, \infty)$, respectively. The partial sum weakly converges to a functional of α-stable process when $\alpha < 2$ and converges to a functional of Brownian motion when $\alpha \geq 2$. When the process is of short-memory and $\alpha < 4$, the autocovariances converge to functionals of $\alpha/2$-stable processes; and if $\alpha \geq 4$, they converge to functionals of Brownian motions. In contrast, when the process is of long-memory, depending on α and β (the parameter that characterizes the long-memory), the autocovariances converge to either (i) functionals of $\alpha/2$-stable processes; (ii) Rosenblatt processes (indexed by β, $1/2 < \beta < 3/4$); or (iii) functionals of Brownian motions. The rates of convergence in these limits depend on both the tail index α and whether or not the linear process is short- or long-memory. Our weak convergence is established on the space of càdlàg functions on $[0, 1]$ with either (i) the J_1 or the M_1 topology (Skorokhod, 1956); or (ii) the weaker form S topology (Jakubowski, 1997). Some statistical applications are also discussed.

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Keywords: GARCH(1,1); Heavy tail; Linear process; Long memory; Rosenblatt process; Short memory

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1. Introduction

A large number of empirical studies show that many financial data series, such as exchange rate returns and stock indices, often exhibit the following non-standard features (see, for instance [40,5]):

(1) Non-gaussianity: the frequency of large and small values (relative to the range of the data) is rather high, suggesting that the data do not come from a normal, but from a heavy-tailed distribution;
(2) Stochastic or time varying volatility: variance changes over time, with alternating phases of high and low volatility;
(3) Long-memory dependence: a slow decay of the autocorrelation function.[1]

Amongst the various models proposed, the generalized autoregressive conditional heteroscedasticity (GARCH) model is one of the most popular ones. Specifically, consider

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \omega + \sum_{i=1}^{r} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} b_j \sigma_{t-j}^2, \tag{1}$$

where $\omega > 0$ and $\{\eta_t\}$ is a sequence of i.i.d. symmetric random variables with unit variance. Under some regularity conditions $\{\varepsilon_t\}$ has a regularly-varying tail probability, which can be used to capture the heavy-tail properties of $\{\varepsilon_t\}$. See, for instance, [40,8].

The GARCH process $\{\varepsilon_t\}$ given by (1) is often β-mixing (see [16]), which is inadequate to account for the strong dependence of the data. To capture the long-memory feature, Baillie et al. [5] proposed a fractional autoregressive integrated moving average (ARFIMA)–GARCH model. This model has been extensively studied. For instance, Baillie et al. [5] used it to model the monthly post-Word War II consumer price index inflation series of 10 different countries. Ling and Li [37] considered the asymptotic properties of the maximum likelihood estimate. Beran and Feng [11] considered a local polynomial estimation of semiparametric models with ARFIMA–GARCH noises. Ling [36] studied the adaptive estimation and applied this model to analyze the US consumer price index inflation series. See [36] and the references therein. However, all these papers only study the long-memory feature but not the heavy-tail feature. A more general model that captures both long-memory and the heavy-tail feature is a linear process with GARCH noises given by

$$u_t = \sum_{l=0}^{\infty} d_l \varepsilon_{t-l}, \tag{2}$$

where $d_0 = 1$ and $\{\varepsilon_t\}$ is a GARCH(r,s) process defined in (1). This paper focuses on the prevalent special case of GARCH(1,1) with $\omega > 0$, $a \geq 0$, $b \geq 0$, such that

$$\varepsilon_t = \sigma_t \eta_t \quad \text{and} \quad \sigma_t^2 = \omega + a\sigma_{t-1}^2 + b\varepsilon_{t-1}^2. \tag{3}$$

This paper is to study the short- and long-memory linear processes in (3) with GARCH(1,1) noises generated by model (4). The functional limit distributions (FLD) of the partial sum and the sample autocovariances are derived when the tail index α is in $(0, 2)$, equal to 2, and in

[1] Mikosch and Stărică [40] also suggested the long-memory dependence of the absolute or squared values. Unfortunately, this phenomenon is not considered in this paper.

$(2, \infty)$, respectively. The partial sum weakly converges to a functional of α-stable process when $\alpha < 2$ and converges to a functional of Brownian motion when $\alpha \geq 2$. When the process is of short-memory and $\alpha < 4$, the autocovariances converge to functionals of $\alpha/2$-stable processes; and if $\alpha \geq 4$, they converge to functionals of Brownian motions. In contrast, when the process is of long-memory, depending on α and β (the parameter that characterizes the long-memory), the autocovariances converge to either (i) functionals of $\alpha/2$-stable processes; (ii) Rosenblatt processes (indexed by β, $1/2 < \beta < 3/4$); or (iii) functionals of Brownian motions. The rates of convergence in these limits depend on both the tail index α and whether or not the linear process is short- or long-memory. Our weak convergence is established on the space of càdlàg functions on $[0, 1]$, $D[0, 1]$, with either (i) the J_1 or the M_1 topology [46]; or (ii) the weaker form S topology [31].

The limit distributions of heavy-tailed linear processes generated by i.i.d. noises have been extensively studied. See, for instance, [1,20,21,32,3,29,27,48,6,43]. To the best of our knowledge, the FLD for heavy-tailed linear processes with GARCH noise are new. Due to the dependence among the GARCH noises, the techniques we use are somewhat different from that for i.i.d. noises, and the cross product terms related to $\varepsilon_t \varepsilon_{t-j}$, $j > 0$ do not vanish asymptotically. As one can see below, our limit distributions, when $\alpha < 2$, depend on an infinite number of point processes, and they somewhat differ from those in the previous studies, which confine the attention to i.i.d. noises.

This paper is organized as follows. Section 2 gives model assumptions. The main results are given in Section 3 while Section 4 gives the proofs. Some statistical applications are discussed in Section 5. Throughout the paper, $\nu^+ = \max(\nu, 0)$, $\nu^- = \max(-\nu, 0)$, $o(1)$ ($o_P(1)$) denotes a series of numbers (random numbers) converging to zero (in probability); $O(1)$ ($O_P(1)$) denotes a series of numbers (random numbers) that are bounded (in probability); when two sequences a_n and b_n are of the same order, we denote $a_n \sim b_n$; $\xrightarrow{P}$ and $\xrightarrow{\mathcal{L}}$ denote convergence in probability and in distribution, respectively; and $\xrightarrow{f.d.d.}$ denotes convergence of finite-dimension distribution. $\xRightarrow{A}$ denotes the weak convergence under A topology, where $A = J_1, M_1, S$. $W(\cdot)$ stands for a standard Brownian motion. $C < \infty$ denotes a positive constant that takes different values in different places.

2. Model assumptions and preliminaries

Throughout, we impose the following 3 assumptions on model (3):

Assumption 2.1. $E \log(a + b\eta_1^2) < 0$. □

Assumption 2.2. There exists a $k_0 > 0$ such that $E(a + b\eta_1^2)^{k_0} \geq 1$ and $E\Big[(a + b\eta_1^2)^{k_0} \log^+(a + b\eta_1^2)\Big] < \infty$, where $\log^+(x) = \max\{\log(x), 0\}$. □

Assumption 2.3. The density of η_1 is positive in a neighborhood of zero.[2] □

Under these assumptions, there exists a constant $\alpha > 0$ such that

$$E\left(a + b\eta_1^2\right)^{\alpha/2} = 1. \tag{4}$$

[2] Assumption 2.3 can be weakened as the distribution of F of η_1 is a mixture of an absolutely continuous component with respect to the Lebesgue measure λ on $\mathcal{R}$ and Dirac masses at some points $\mu_i \in \mathcal{R}$, $i = 1, \ldots, N$. See [23].

See, for instance, [34]. Further, when $\alpha \in (2, \infty)$, the GARCH process $\{\varepsilon_t\}$ has a finite variance. When $\alpha = 2$, $\{\varepsilon_t\}$ is called the IGARCH process and it has an infinite variance. The IGARCH process is particularly interesting as, in fitting the log return of asset price to a GARCH(1,1) model, it is often reported that the estimated $a + b$ is close to unity. On the other hand, when $\alpha \in (0, 2]$, $\{\varepsilon_t\}$ also has an infinite variance. Goldie [25] shows there exists a positive constant $c_0^{(\alpha)}$ such that as $x \to \infty$,

$$P\left(\sigma_1^2 > x\right) = c_0^{(\alpha)} x^{-\alpha/2}\{1 + o(1)\}, \quad \text{which gives that} \tag{5}$$

$$\begin{aligned} P(|\varepsilon_1| > x) &= P(|\sigma_1\eta_1| > x) = \left(E|\eta_1|^\alpha\right) P\left(\sigma_1^2 > x^2\right) \\ &= \left(E|\eta_1|^\alpha\right) c_0^{(\alpha)} x^{-\alpha}\{1 + o(1)\}, \end{aligned} \tag{6}$$

provided $E|\eta_1|^\alpha < \infty$. See also [14]. Hence $P(|\varepsilon_1| > x)$ is regularly varying with index α, that is, $\lim_{t\to\infty} P(|\varepsilon_1| > tx)/P(|\varepsilon_1| > t) = x^{-\alpha}$ for $x > 0$. It follows that $\lim_{n\to\infty} nP\left(|\varepsilon_1| > a_n^{(\alpha)}\right) = 1$, where

$$a_n^{(\alpha)} := \left(c_0^{(\alpha)} E|\eta_1|^\alpha n\right)^{1/\alpha}. \tag{7}$$

When no ambiguity arises, write $c_0 = c_0^{(\alpha)}$, $a_n = a_n^{(\alpha)}$. By Theorem 4.1 in [25],

$$c_0^{(\alpha)} := \frac{E\left(\left[\omega + \left(a + b\eta_1^2\right)\sigma_1^2\right]^{\alpha/2} - \left[\left(a + b\eta_1^2\right)\sigma_1^2\right]^{\alpha/2}\right)}{\alpha/2E\left((a + b\eta_1^2)^{\alpha/2}\log^+(a + b\eta_1^2)\right)}. \tag{8}$$

For $\alpha = 2$, since $E\eta_1^2 = 1$, we may write $a_n^{(2)} = \sqrt{c_0^{(2)} n}$.

The α-stable limits derived in the next section are expressed as infinite series of the points of Poisson processes. Following [18,19], for any positive integers (l, H), define an $(H + 1)$-dimensional random vector:

$$\boldsymbol{X}_{t,l,H} := (\varepsilon_{t-l}, \varepsilon_{t-l-1}, \ldots, \varepsilon_{t-l-H}) =: \left(X_{t,l}^{(0)}, X_{t,l}^{(1)}, \ldots, X_{t,l}^{(H)}\right). \tag{9}$$

By Theorem 2.8 in [19],[3] there exists a Poisson process $\sum_{i=1}^\infty \delta_{P_i} := \sum_{i=1}^\infty \delta_{P_i(\alpha)}$ defined on $\mathcal{R}_+$ with mean measure $v(dy) = \Upsilon\alpha y^{-\alpha-1}\,dy$, and a sequence of i.i.d. point processes $\{\sum_{j=1}^\infty \delta_{\boldsymbol{Q}_{ij,H}}\} := \{\sum_{j=1}^\infty \delta_{\boldsymbol{Q}_{ij,H}(\alpha)}\}$ which is independent of $\{P_i\}$,

$$\sum_{t=1}^n \delta_{\boldsymbol{X}_{t,l,H}/a_n} \xrightarrow{\mathcal{L}} \sum_{i=1}^\infty \sum_{j=1}^\infty \delta_{P_i \boldsymbol{Q}_{ij,H}}, \tag{10}$$

where $\boldsymbol{Q}_{ij,H} = \left(Q_{ij}^{(0)}, Q_{ij}^{(1)}, \ldots, Q_{ij}^{(H)}\right)$ with a common distribution equal to:

$$\lim_{k\to\infty}\lim_{n\to\infty} P\left(\sum_{|t|\le k} \delta_{\boldsymbol{X}_{t,l,H}/(\sup_{|t|\le k}|\boldsymbol{X}_{t,l,H}|)} \in \cdot \,\middle|\, \sup_{1\le t\le k} |\boldsymbol{X}_{t,l,H}| \le a_n \le |\boldsymbol{X}_{0,l,H}|\right), \tag{11}$$

[3] See also Theorem 3.1 in [40].

in which the norm $\left|\boldsymbol{X}_{t,l,H}\right| = \max_{0\le h\le H}\left|X_{t,l}^{(h)}\right|$. Further, the extremal index,[4]

$$\Upsilon = \lim_{k\to\infty}\lim_{n\to\infty} P\left\{\sum_{t=1}^{k}\left|\boldsymbol{X}_{t,l,H}\right| \le a_n, \left||\boldsymbol{X}_{0,l,H}| > a_n\right.\right\}. \tag{12}$$

As pointed out by a referee, using the same assumptions as ours and considering $\alpha \in (0, 2)$ or $\alpha \in (0, 2) \cup (2, 4)$, Proposition 5 in [7] expressed the limiting results in terms of characteristic functions (contrast to Lemmas 4.1(a) and 4.3(a)). The major advantage of their classical blocking and mixing techniques over the point process approach is, by controlling clustering of big values, one may calculate the parameters of the stable limit in terms of quantities of the finite-dimensional distributions of the underlying process. The advantages of Bartkiewicz et al.'s [7] approach will be exploited in the future research.

3. Main results

3.1. Partial sum of the short- and long-memory processes

In this subsection, we give the weak convergence of the partial sum of $\{u_t\}$ in (2) when it is a short- or long-memory linear process with a GARCH(1,1) noise $\{\varepsilon_t\}$ in (3).

Theorem 3.1. *Suppose Assumptions 2.1–2.3 hold and $\sum_{l=0}^{\infty}|d_l|^{\gamma} < \infty$ for some $\gamma \le 1, \gamma < \alpha$, α is given by (4).*

$$\text{(a)}\ \frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau\rfloor} u_t \overset{S}{\Longrightarrow} \left(\sum_{l=0}^{\infty} d_l\right)\xi_\alpha(\tau), \quad 0 < \alpha < 2; \tag{13}$$

$$\text{(b)}\ \frac{1}{\sqrt{nc_0\log n}}\sum_{t=1}^{\lfloor n\tau\rfloor} u_t \overset{J_1}{\Longrightarrow} \left(\sum_{l=0}^{\infty} d_l\right)W(\tau), \quad \alpha = 2; \tag{14}$$

$$\text{(c)}\ \frac{1}{\sqrt{n\left(E\sigma_1^2\right)}}\sum_{t=1}^{\lfloor n\tau\rfloor} u_t \overset{J_1}{\Longrightarrow} \left(\sum_{l=0}^{\infty} d_l\right)W(\tau), \quad \alpha > 2; \tag{15}$$

where $\xi_\alpha(\cdot)$ is an α-stable process with $\xi_\alpha(1) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i(\alpha)Q_{ij}^{(0)}(\alpha)$. □

Remark 3.1. Using the point process technique, Davis and Hsing [18] showed

$$\sum_{t=1}^{n}\delta_{\varepsilon_t/a_n} \xrightarrow{\mathcal{L}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{P_i(\alpha)Q_{ij}^{(0)}(\alpha)}, \tag{16}$$

for any $0 < \alpha < 2$. By the continuous mapping theorem, for any $\varrho > 0$ and for $\tau = 1$,

$$a_n^{-1}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t I\left(|\varepsilon_t|/a_n > \varrho\right) \xrightarrow{\mathcal{L}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i(\alpha)Q_{ij}^{(0)}(\alpha) I\left(|P_i(\alpha)Q_{ij}^{(0)}(\alpha)| > \varrho\right)$$
$$=: \xi_\alpha^{(\varrho)}(\tau). \tag{17}$$

[4] See also Remark 2.3 in [19]; or Remark 4.7 in [10].

For any $\epsilon > 0$, $\lim_{\varrho \downarrow 0} \limsup_{n \uparrow \infty} P\left(\left|a_n^{-1} \sum_{t=1}^{n} \left(\varepsilon_t I\left(|\varepsilon_t|/a_n \le \varrho\right) - E\varepsilon_1 I\left(|\varepsilon_1|/a_n \le \varrho\right)\right)\right| > \epsilon\right) = 0$. Recall ε_1 is symmetrically distributed and $E\varepsilon_1/a_n I\left(|\varepsilon_1|/a_n \le 1\right) = 0$. Thus,

$$a_n^{-1} \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t \xrightarrow{\mathcal{L}} \lim_{\varrho \downarrow 0} \xi_\alpha^{(\varrho)}(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i(\alpha) Q_{ij}^{(0)}(\alpha) =: \xi_\alpha(\tau), \quad \text{for } \tau = 1. \tag{18}$$

The convergence of (17)–(18) was generalized to all $\tau \in [0, 1]$ by Basrak, Krizmanić and Segers [9, Theorem 3.4]. More precisely, they showed (17)–(18) hold in $D[0, 1]$ under the M_1 topology, by considering a time-space process N in $[-T, 1] \times [-\varrho, \varrho]^c$, $T > 0$. N is a Poisson process with mean measure $\lambda \times \nu^{(\varrho)}$, λ is the Lebesgue measure and for $x > 0$,

$$\nu^{(\varrho)}([-x, x]^c) := \varrho^{-\alpha} P\left(\varrho \sum_{t=1}^{\infty} |Y_t| I\left(|Y_t| > 1\right) > x, \sup_{-\infty \le j \le -1} |Y_j| \le 1\right), \tag{19}$$

$\{Y_t\}$ is the tail process of $\{\varepsilon_t\}$ (Theorems 2.1 and 2.3 in [9]). Thus,

$$\xi_\alpha^{(\varrho)}(\tau) = \int_0^{\tau+} \int_{[-\varrho,\varrho]^c} r N(ds, dr), \qquad \xi_\alpha(\tau) = \lim_{\varrho \downarrow 0} \xi_\alpha^{(\varrho)}(\tau), \quad \text{for } \tau \in [0, 1]. \quad \square \tag{20}$$

Theorem 3.2. *Suppose Assumptions 2.1–2.3 hold,* $\lim_{n\to\infty} \frac{1}{n^{1-\beta} l(n)} \sum_{j=0}^{n} |d_j| < \infty$, $d_n = O(l(n)/n^\beta)$, $\max\{\frac{1}{\alpha}, \frac{1}{2}\} < \beta \le 1$, *$\alpha$ is given by (4), and $l(n)$ is a slowly varying function.*

$$\text{(a)}\ \frac{1}{n^{1-\beta} l(n) a_n} \sum_{t=1}^{\lfloor n\tau \rfloor} u_t \overset{S}{\Longrightarrow} K \int_{-\infty}^{\tau} X(s, \tau)\, d\xi_\alpha(s), \quad 1 < \alpha < 2; \tag{21}$$

$$\text{(b)}\ \frac{1}{n^{1-\beta} l(n) \sqrt{n c_0 \log n}} \sum_{t=1}^{\lfloor n\tau \rfloor} u_t \overset{J_1}{\Longrightarrow} K \int_{-\infty}^{\tau} X(s, \tau)\, dW(s), \quad \alpha = 2; \tag{22}$$

$$\text{(c)}\ \frac{1}{n^{1-\beta} l(n) \sqrt{n \left(E\sigma_1^2\right)}} \sum_{t=1}^{\lfloor n\tau \rfloor} u_t \overset{J_1}{\Longrightarrow} K \int_{-\infty}^{\tau} X(s, \tau)\, dW(s), \quad \alpha > 2; \tag{23}$$

where $K = \lim_{n\to\infty} \frac{1}{n^{1-\beta} l(n)} \sum_{j=0}^{n} d_j$, *and*

$$X(s, \tau) := \begin{cases} \left((s - \tau)^-\right)^{1-\beta} - \left(s^-\right)^{1-\beta}, & \beta < 1, \\ I\left(0 \le s \le \tau\right), & \beta = 1; \end{cases} \tag{24}$$

$$\int_{-\infty}^{\tau} X(s, \tau)\, d\xi_\alpha(s) := \lim_{T\uparrow\infty} \lim_{\varrho\downarrow 0} \int_{-T}^{\tau+} \int_{[-\varrho,\varrho]^c} X(s, \tau) r N(ds, dr); \tag{25}$$

in which the random measure $N(ds, dr)$ is defined around (19). $\square$

Remark 3.2. Consider an $ARFIMA(p, 1 - \iota, q)$ model, $p, q < \infty$, $\max\{\frac{1}{\alpha}, \frac{1}{2}\} < \iota \le 1$:

$$\phi(B)(1 - B)^{1-\iota} u_t = \theta(B)\varepsilon_t, \tag{26}$$

where B is the lag operator and $\{\varepsilon_t\}$ is a GARCH(1,1) noise specified in (3).

(a) When $\iota = 1$, the conclusions of Theorem 3.1 hold with $\sum_{l=0}^{\infty} d_l = \frac{\theta(1)}{\phi(1)}$.

(b) When $\max\{1/\alpha, 1/2\} < \iota < 1$, the conclusions of Theorem 3.2 hold with $K = \frac{\theta(1)}{\phi(1)\Gamma(1-\iota)}$, where $\Gamma(\cdot)$ is the gamma function.[5] □

Remark 3.3. Theorem 3.2(a) extends Theorem 3.9 of [6] to the S-convergence of a long-memory linear process of GARCH(1,1) noises, where $1 < \alpha < 2$. To the best of our knowledge, the result in Theorem 3.2(b), where $\alpha = 2$, is new; while Theorem 3.2(c), where $\alpha > 2$, is a special case of Theorem 2 of [49]. □

3.2. Sample autocovariance of the short- and long-memory processes

In this subsection, we study the weak convergence of the sample autocovariance of $\{u_t\}$. For Theorem 3.1 (and Theorem 3.3), let $p = \gamma$; and for Theorem 3.2 (and Theorem 3.4), let $1 \le 1/\beta < p < \min\{2, \alpha\}$. In either case, for $0 \le k < \infty$,

$$\sum_{l=0}^{\infty} |d_l|^p < \infty \Longrightarrow \sum_{l=0}^{\infty} |d_l d_{l+k}|^{p/2} \le \sum_{l=0}^{\infty} \left(|d_l|^p + |d_{l+k}|^p\right)/2 < \infty, \tag{27}$$

where $p < \min\{2, \alpha\}$. Thus we define

$$\gamma_n^{(k)} = \begin{cases} 0, & 0 < \alpha < 2, \\ \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right) c_0 \log n, & \alpha = 2, \\ \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right) (E\sigma_1^2), & \alpha > 2. \end{cases} \tag{28}$$

Denote $P_i = P_i(\alpha)$ and $Q_{ij}^{(h)} = Q_{ij}^{(h)}(\alpha), h \ge 0$. For $0 < \alpha < 4$ in Theorems 3.3 and 3.4, the limit involves a $\alpha/2$-stable process, $S_{\alpha/2}^{(h)}(\cdot)$. When $0 < \alpha < 2$, or $h \ge 1$,[6]

$$S_{\alpha/2}^{(h)}(1) = \sum_{i,j\ge 1} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}. \tag{29}$$

When $2 \le \alpha < 4$ and $h = 0$, $S_{\alpha/2}^{(0)}(1)$ is the distributional limit of

$$\sum_{i,j\ge 1} P_i^2 Q_{ij}^{(0)2} I\left(\varrho < P_i^2 Q_{ij}^{(0)2}\right) - \int_{\sqrt{\varrho}<x\le 1} x^2 \mu(dx), \quad \alpha = 2; \tag{30}$$

$$\sum_{i,j\ge 1} P_i^2 Q_{ij}^{(0)2} I\left(\varrho < P_i^2 Q_{ij}^{(0)2}\right) - \int_{\sqrt{\varrho}<x} x^2 \mu(dx), \quad 2 < \alpha < 4; \tag{31}$$

as $\varrho \downarrow 0$, where by (6)–(7), $\mu(dx) = \alpha x^{-\alpha-1}dx$.

Remark 3.4. Deriving the limit behavior of the sample autocovariance of a sequence not necessarily a m.d.s., Theorem 3.5 in [19] did not consider the case $\alpha = 2$. In (30) where

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[5] See, for instance, [28,35].

[6] For $2 \le \alpha < 4$, $\varepsilon_t\varepsilon_{t-h} = \sigma_{t-h}|\eta_{t-h}|\sigma_t sign(\eta_{t-h})\eta_t$, which is symmetrically distributed, as $sign(\eta_{t-h})\eta_t$ is independent of $\sigma_{t-h}|\eta_{t-h}|\sigma_t$. Thus, the points $\sum_{j\ge 1} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}$ are summable.

$\alpha = 2$, the centering constant is $E\varepsilon_1^2 I(|\varepsilon_1| \le a_n) = c_0 \log n$. On the other hand, in (31) where $2 \le \alpha < 4$, as in [19], the centering constant is $E\varepsilon_1^2 = E\sigma_1^2$. See also Lemmas 4.2 and 4.3. □

Theorem 3.3. *Suppose the assumptions in Theorem 3.1 hold. For $0 \le k < \infty$,*

$$\text{(a)}\ \frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau \rfloor}(u_t u_{t-k} - \gamma_n^{(k)}) \overset{S}{\Longrightarrow} Z_{\alpha/2}^{(k)}(\tau), \quad 0 < \alpha < 4; \tag{32}$$

$$\text{(b)}\ \frac{1}{\sqrt{n \log n}}\sum_{t=1}^{\lfloor n\tau \rfloor}(u_t u_{t-k} - \gamma_n^{(k)}) \overset{J_1}{\Longrightarrow} K_1^{(k)} W(\tau), \quad \alpha = 4; \tag{33}$$

$$\text{(c)}\ \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\tau \rfloor}(u_t u_{t-k} - \gamma_n^{(k)}) \overset{J_1}{\Longrightarrow} K_2^{(k)} W(\tau), \quad \alpha > 4; \tag{34}$$

where $Z_{\alpha/2}^{(k)}(\cdot)$ is a $\alpha/2$-stable process with

$$Z_{\alpha/2}^{(k)}(1) = \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right) S_{\alpha/2}^{(0)}(1) + \sum_{h=1}^{\infty}\left(\sum_{l=0}^{\infty} d_l d_{l+h+k} + \sum_{l=0}^{\infty} d_l d_{l+|h-k|}\right) S_{\alpha/2}^{(h)}(1), \tag{35}$$

$$\begin{aligned} K_1^{(k)2} &= \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right)^2 c_0 E\left(\eta_1^4 - 1\right)\left(\frac{1-a}{1-a-b}\right)^2 \\ &\quad + \sum_{h=1}^{\infty}\left[\left(\sum_{l=0}^{\infty} d_l d_{l+h+k} + \sum_{l=0}^{\infty} d_l d_{l+|h-k|}\right)^2 (a+b)^{(h-1)}\right] c_0 \left(a + bE\eta_1^4\right), \end{aligned} \tag{36}$$

$$\begin{aligned} K_2^{(k)2} &= \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right)^2 (E\sigma_1^4) E\left(\eta_1^4 - 1\right)\left(\frac{1-a}{1-a-b}\right)^2 \\ &\quad + \sum_{h=1}^{\infty}\left[\left(\sum_{l=0}^{\infty} d_l d_{l+h+k} + \sum_{l=0}^{\infty} d_l d_{l+|h-k|}\right)^2 E\left(\varepsilon_1^2 \varepsilon_{1-h}^2\right)\right], \end{aligned} \tag{37}$$

in which for $\alpha > 4$, with $A_1 = a + b\eta_0^2$ and defining $\pi := a + b$,[7]

$$E\sigma_1^4 = \frac{\omega^2(1+\pi)}{(1-\pi)(1-EA_1^2)},$$

$$E\left(\varepsilon_1^2 \varepsilon_{1-h}^2\right) = \left(E\sigma_1^4\right)\left(a + bE\eta_1^4\right)\pi^{h-1} + \frac{\omega^2\left(1-\pi^h\right)}{(1-\pi)^2}, \quad h \ge 1.$$

Theorem 3.4. *Suppose the assumptions in Theorem 3.2 hold. For $0 \le k < \infty$,*

(a) *If $1 < \alpha \le 2$, or $2 < \alpha < 4$ with $\beta > 1 - 1/\alpha$,*

[7] When $\alpha > 4$ and $\pi = 0$, write $\gamma(s) = \omega^2 \sum_{l=0}^{\infty} d_l d_{l+s}$ and observe that $\gamma(s) = \gamma(-s)$, $K_2^{(k)2} = E(\eta_1^4 - 3)\gamma^2(k) + \sum_{h=-\infty}^{\infty}\left(\gamma^2(h) + \gamma(h+k)\gamma(h-k)\right)$, which is the asymptotic variance of the sample autocovariance of order k, when $\{\varepsilon_t\}$ is an i.i.d. sequence with finite 4th moment. See, for instance, Proposition 7.3.1 in [15].

$$\frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor}(u_t u_{t-k}-\gamma_n^{(k)}) \stackrel{S}{\Longrightarrow} Z_{\alpha/2}^{(k)}(\tau); \tag{38}$$

(b) *If* $2<\alpha\le 4$ *with* $\beta<1-1/\alpha$, *or* $\alpha>4$ *with* $\beta<3/4$,

$$\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=1}^{\lfloor n\tau\rfloor}(u_t u_{t-k}-\gamma_n^{(k)}) \stackrel{J_1}{\Longrightarrow} (E\sigma_1^2)U_\beta(\tau); \tag{39}$$

(c) *If* $\alpha=4$ *with* $\beta>3/4$,

$$\frac{1}{\sqrt{n\log n}}\sum_{t=1}^{\lfloor n\tau\rfloor}(u_t u_{t-k}-\gamma_n^{(k)}) \stackrel{J_1}{\Longrightarrow} K_1^{(k)}W(\tau); \tag{40}$$

(d) *If* $\alpha>4$ *with* $\beta>3/4$,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\tau\rfloor}(u_t u_{t-k}-\gamma_n^{(k)}) \stackrel{J_1}{\Longrightarrow} K_2^{(k)}W(\tau); \tag{41}$$

where $Z_{\alpha/2}^{(k)}(\cdot)$ *is defined as in* (35), $U_\beta(\cdot)$ *is a Rosenblatt process defined as:*

$$U_\beta(\tau)=2\int_{s_1<s_2<\tau}\left[\int_0^\tau \left((r-s_1)^+\right)^{-\beta}\left((r-s_2)^+\right)^{-\beta}dr\right]W(ds_1)W(ds_2), \tag{42}$$

and $K_1^{(k)2}$ *and* $K_2^{(k)2}$ *are defined as in* (36)–(37). □

Remark 3.5. Consider model (26) in Remark 3.2.

(a) When $\iota=1$, the conclusions of Theorem 3.3 hold with $\sum_{j=0}^{\infty}d_jB^j=\frac{\theta(B)}{\phi(B)}$; and $\sum_{j=0}^{\infty}d_jd_{j+s}$, $s\ge 0$ being defined accordingly.

(b) When $\max\{1/\alpha,1/2\}<\iota<1$, the conclusions of Theorem 3.4 hold with $\sum_{j=0}^{\infty}d_jB^j=\frac{\theta(B)(1-B)^{(\iota-1)}}{\phi(B)}$, where

$$(1-B)^{(\iota-1)}=\sum_{j=0}^{\infty}\psi_jB^j,\quad \text{in which } \psi_0=1,\ \psi_j=\prod_{i=1}^{j}\frac{i-\iota}{i},\ j\ge 1;$$

and $\sum_{j=0}^{\infty}d_jd_{j+s}$, $s\ge 0$ being defined accordingly.[8] □

4. Proofs

Define $\sum_{t=1}^{\lfloor -n\tau\rfloor}=\sum_{t=\lfloor -n\tau\rfloor}^{0}$ for $\tau>0$. Lemma 4.1 for the partial sum of $\{\varepsilon_t\}$ is essentially Theorem 2.1(b), 2.1(c) and 2.2(a) in [17], as well as Theorem 18.3 in [13]. It plays a crucial role in establishing the FLD of $\{u_t\}$.

Lemma 4.1. *Suppose Assumptions 2.1–2.3 hold with* α *given by* (4).

$$\text{(a)}\ \frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t \stackrel{S}{\Longrightarrow} \xi_\alpha(\tau),\quad 0<\alpha<2; \tag{43}$$

[8] The detailed formulae can be found in pp. 172–173 in [47] and pp. 47–48 in [42].

$$\text{(b)}\ \frac{1}{\sqrt{nc_0 \log n}} \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t \overset{J_1}{\Longrightarrow} W(\tau), \quad \alpha = 2; \tag{44}$$

$$\text{(c)}\ \frac{1}{\sqrt{n(E\sigma_1^2)}} \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t \overset{J_1}{\Longrightarrow} W(\tau), \quad \alpha > 2; \tag{45}$$

where $\xi_\alpha(\cdot)$ is an α-stable process with $\xi_\alpha(1)$ defined as in Theorem 3.1. □

Proof. We first show (43) where $0 < \alpha < 2$. Let $S_n(\tau) = \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t / a_n$. By the arguments for Theorem 2.1(b) in [17] (see also Lemmas A.5–A.6 there),

$$S_n(\tau) \overset{f.d.d.}{\longrightarrow} \xi_\alpha(\tau). \tag{46}$$

Let $S_n(\tau) = \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t I(\sigma_t \le a_n)/a_n + \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t I(\sigma_t > a_n)/a_n =: S_{1n}(\tau) + S_{2n}(\tau)$. For all n, $\{\varepsilon_t I(\sigma_t \le a_n)/a_n\}$ is a m.d.s. and $\sup_n \sup_{\tau\in[-T,1]} E|S_{1n}(\tau)| < \infty$. By Doob's inequality,

(a) $\{\max_{-T\le\tau\le1} |S_{1n}(\tau)|\}$ is stochastically bounded.
(b) For any $a < b, a, b \in \mathcal{R}$, $\{N^{a,b}(S_{1n})\}$ is stochastically bounded,

where $N^{a,b}(S_{1n})$ is the number of up-crossings of $[a, b]$ by the process S_{1n}. Similarly define $N^{a,b}(S_{2n})$ and $N^{a,b}(S_n)$. Next let $\varrho = \min\{\alpha - \epsilon, 1\}$, $0 < \epsilon < \alpha$,

$$\max_n E\left[\max_{-T\le\tau\le1} |S_{2n}(\tau)|\right]^{\varrho} \le \max_n a_n^{-\varrho} \sum_{\lfloor -nT \rfloor \le t \le n} E\left[|\varepsilon_t|^{\varrho} I(\sigma_t > a_n)\right] < \infty,$$

$$E\left[N^{a,b}(S_{2n})\right] \le E\left[\sum_{\lfloor -nT \rfloor \le t \le n} I(\sigma_t > a_n)\right] = (\lfloor nT \rfloor + n + 1)\, P(\sigma_1 > a_n) < \infty.$$

Thus, $\{\max_{-T\le\tau\le1} |S_{2n}(\tau)|\}$ and $\{N^{a,b}(S_{2n})\}$ are also stochastically bounded. Since

$$\max_{-T\le\tau\le1} |S_n(\tau)| \le \max_{-T\le\tau\le1} |S_{1n}(\tau)| + \max_{-T\le\tau\le1} |S_{2n}(\tau)|,$$
$$N^{a,b}(S_n) \le N^{a,b}(S_{1n}) + N^{a,b}(S_{2n}),$$

it follows that $\{\max_{-T\le\tau\le1} |S_n(\tau)|\}$ and $\{N^{a,b}(S_n)\}$ are also stochastically bounded. By Theorem 3.2 in [31], $\{S_n(\tau)\}$ is relatively compact and (43) is proved. Next, we show (44) where $\alpha = 2$. Let $\varepsilon_{t1} = \varepsilon_t I(\sigma_t \le \sqrt{nc_0 \log\log n})$, $\varepsilon_{t2} = \varepsilon_t I(\sigma_t > \sqrt{nc_0 \log\log n})$ and $k_n = \sqrt{nc_0 \log n}$. Now define $S_{1n}(\tau) = \frac{1}{k_n} \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_{t1}$ and $S_{2n}(\tau) = \frac{1}{k_n} \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_{t2}$. By arguments for (6.3) in [17],

$$S_{1n}(\tau) \overset{J_1}{\Longrightarrow} W(\tau). \tag{47}$$

On the other hand, for any $0 < \epsilon < 2$,

$$E\left[\left|\sup_{-T\le\tau\le1} S_{2n}(\tau)\right|^{2-\epsilon}\right] \le \frac{1}{k_n^{2-\epsilon}} \sum_{t=\lfloor -nT \rfloor}^{n} E\,|\varepsilon_{t2}|^{2-\epsilon} \le C \frac{(n \log\log n)^{1-\epsilon/2}}{(n \log n)^{1-\epsilon/2}} \to 0. \tag{48}$$

It follows that $\sup_{-T\le\tau\le1} S_{2n}(\tau) \overset{P}{\longrightarrow} 0$ and thus (44) holds. Finally, when $\alpha > 2$, $\sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_t$ is a square integrable martingale. Eq. (45) follows by the martingale functional central limit theorem. See Theorem 18.3 in [13]. □

Proof of Theorem 3.1. Write

$$u_t = \sum_{l=0}^{\infty} d_l \varepsilon_{t-l} = \sum_{l=0}^{H} d_l \varepsilon_{t-l} + \sum_{l=H+1}^{\infty} d_l \varepsilon_{t-l} =: I_{1t}(H) + I_{2t}(H). \tag{49}$$

We first show (13). By Lemma 4.1(a), for any integer $0 \le l \le H$, $\frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_{t-l} \overset{S}{\Longrightarrow} \xi_\alpha(\tau)$. Since addition is sequentially continuous with respect to the S topology (see Theorem 2.13 in [31]), it follows that

$$\frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor} I_{1t}(H) = \sum_{l=0}^{H} d_l \left(\frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_{t-l} \right) \overset{S}{\Longrightarrow} \left(\sum_{l=0}^{H} d_l \right) \xi_\alpha(\tau). \tag{50}$$

By Theorem 3.2 in [13], it remains to show for any $\varrho > 0$,

$$\lim_{H\to\infty} \limsup_{n\to\infty} P\left(\sup_{0\le\tau\le 1} \left| \frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor} I_{2t}(H) \right| > \varrho \right) = 0, \quad \text{and} \tag{51}$$

$$\text{as } H\to\infty, \left(\sum_{l=0}^{H} d_l \right) \xi_\alpha(\tau) \overset{P}{\longrightarrow} \left(\sum_{l=0}^{\infty} d_l \right) \xi_\alpha(\tau). \tag{52}$$

Given $\sum_{l=0}^{\infty} |d_l| < \infty$, (52) results. It remains to show (51). Write

$$\begin{aligned} \frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor} I_{2t}(H) &= \frac{1}{a_n}\sum_{l=H+1}^{\infty} d_l \sum_{t=1}^{\lfloor n\tau \rfloor} \varepsilon_{t-l} I\left(\sigma_{t-l} \le a_n\right) \\ &\quad + \frac{1}{a_n}\sum_{t=1}^{\lfloor n\tau \rfloor}\sum_{l=H+1}^{\infty} d_l \varepsilon_{t-l} I\left(\sigma_{t-l} > a_n\right) \\ &=: \Sigma_{1n}(\tau) + \Sigma_{2n}(\tau). \end{aligned} \tag{53}$$

Note that for any fixed l, $\{\varepsilon_{t-l} I\left(\sigma_{t-l} \le a_n\right)\}$ is a m.d.s. By Doob's inequality,

$$E\left| \sup_{0\le\tau\le 1} \frac{1}{a_n}\sum_{t=1}^{[n\tau]} \varepsilon_{t-l} I\left(\sigma_{t-l} \le a_n\right) \right|^2 \le 4E\left| \frac{1}{a_n}\sum_{t=1}^{n} \varepsilon_{t-l} I\left(\sigma_{t-l} \le a_n\right) \right|^2 \to \frac{4\alpha}{2-\alpha}, \tag{54}$$

as $n \to \infty$, where the last limit follows by the uncorrelatedness of ε_{t-l}'s and Karamata's theorem. By (54), since $\sum_{l=0}^{\infty} |d_l| < \infty$, we have

$$\begin{aligned} \left[E\left| \sup_{0\le\tau\le 1} \Sigma_{1n}(\tau) \right|^2 \right]^{1/2} &\le \sum_{l=H+1}^{\infty} |d_l| \left[E\left| \sup_{0\le\tau\le 1} \frac{1}{a_n}\sum_{t=1}^{[n\tau]} \varepsilon_{t-l} I\left(\sigma_{t-l} \le a_n\right) \right|^2 \right]^{1/2} \\ &\le 2\sqrt{\alpha(2-\alpha)^{-1}} \sum_{l=H+1}^{\infty} |d_l| \to 0, \end{aligned} \tag{55}$$

as $n \to \infty$ followed by $H \to \infty$. Thus $\sup_{0\le\tau\le 1} \Sigma_{1n}(\tau) \overset{P}{\longrightarrow} 0$. Let γ be that specified in the Theorem.

$$E\left|\sup_{0\le\tau\le 1}\Sigma_{2n}(\tau)\right|^{\gamma} \le \sum_{t=1}^{n}\sum_{l=H+1}^{\infty}|d_l|^{\gamma} E\left|\frac{1}{a_n^{\gamma}}|\varepsilon_{t-l}|^{\gamma} I\left(\sigma_{t-l}>a_n\right)\right|$$
$$\le \alpha(\alpha-\gamma)^{-1}\sum_{l=H+1}^{\infty}|d_l|^{\gamma}\to 0,$$

as $n\to\infty$ followed by $H\to\infty$. Thus, $\sup_{0\le\tau\le 1}\Sigma_{2n}(\tau)\xrightarrow{P}0$. And (13) is proved. Next, we show (14). Let $k_n=\sqrt{nc_0\log n}$. By Lemma 4.1(b), for any fixed integer $H>0$,

$$\frac{1}{k_n}\left(\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t, \sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-1},\ldots,\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-H}\right)\overset{J_1}{\Longrightarrow}\left(W(\tau),W(\tau),\ldots,W(\tau)\right). \tag{56}$$

Given $\sum_{l=0}^{\infty}|d_l|<\infty$,

$$\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}I_{1t}(H)\overset{J_1}{\Longrightarrow}\left(\sum_{l=1}^{H}d_l\right)W(\tau)\xrightarrow{P}\left(\sum_{l=1}^{\infty}d_l\right)W(\tau), \tag{57}$$

as $n\to\infty$ followed by $H\to\infty$. Write

$$\frac{1}{k_n}\sum_{t=1}^{[n\tau]}I_{2t}(H)=\frac{1}{k_n}\sum_{l=H+1}^{\infty}d_l\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-l}I\left(\sigma_{t-l}\le k_n\right)+\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{l=H+1}^{\infty}d_l\varepsilon_{t-l}I\left(\sigma_{t-l}>k_n\right).$$

Using the same argument for (51), $\sup_{0\le\tau\le 1}\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}I_{2t}(H)\xrightarrow{P}0$. Thus, Eq. (14) holds. Finally, we turn to (15). Using Lemma 4.1(c) and the same argument for (57),

$$\frac{1}{\sqrt{n\left(E\sigma_1^2\right)}}\sum_{t=1}^{\lfloor n\tau\rfloor}I_{1t}(H)\overset{J_1}{\Longrightarrow}\left(\sum_{l=1}^{\infty}d_l\right)W(\tau).$$

For any fixed l, $\{\varepsilon_{t-l}\}$ is a square integrable m.d.s. By Doob's inequality,

$$\left[E\left|\sup_{0\le\tau\le 1}\frac{1}{\sqrt{n\left(E\sigma_1^2\right)}}\sum_{t=1}^{\lfloor n\tau\rfloor}I_{2t}(H)\right|^2\right]^{1/2}\le C\sum_{l=H+1}^{\infty}|d_l|\to 0, \tag{58}$$

as $H\to\infty$. Thus, (15) results and the proof is complete. □

Proof of Theorem 3.2. First consider (21). Let $\varepsilon_{i1}=\varepsilon_i I(\sigma_i\le a_n)$, $\varepsilon_{i2}=\varepsilon_i I(\sigma_i>a_n)$.

$$\sum_{t=1}^{\lfloor n\tau\rfloor}u_t=\sum_{i=-\infty}^{\lfloor n\tau\rfloor}\sum_{t=i\vee 1}^{\lfloor n\tau\rfloor}d_{t-i}\varepsilon_i=\sum_{i=-\infty}^{-\lfloor nT\rfloor-1}\sum_{t=1}^{\lfloor n\tau\rfloor}d_{t-i}\varepsilon_i+\sum_{i=-\lfloor nT\rfloor}^{\lfloor n\tau\rfloor}\sum_{t=i\vee 1}^{\lfloor n\tau\rfloor}d_{t-i}\varepsilon_i$$
$$=\sum_{i=\lfloor nT\rfloor+1}^{\infty}\sum_{t=1}^{\lfloor n\tau\rfloor}d_{t+i}\varepsilon_{-i1}+\sum_{i=\lfloor nT\rfloor+1}^{\infty}\sum_{t=1}^{\lfloor n\tau\rfloor}d_{t+i}\varepsilon_{-i2}+\sum_{i=-\lfloor nT\rfloor}^{\lfloor n\tau\rfloor}\sum_{t=i\vee 1}^{\lfloor n\tau\rfloor}d_{t-i}\varepsilon_i$$
$$=:\sum_{j=1}^{3}I_{jn}(T,\tau). \tag{59}$$

Eq. (21) follows by proving:

$$\sup_{0\le\tau\le 1}\frac{1}{n^{1-\beta}l(n)a_n}\left|I_{1n}(T,\tau)+I_{2n}(T,\tau)\right|\xrightarrow{P}0,\quad\text{and}\tag{60}$$

$$\frac{1}{n^{1-\beta}l(n)a_n}I_{3n}(T,\tau)\overset{S}{\Longrightarrow}K\int_{-\infty}^{\tau}X(s,\tau)d\xi_\alpha(s).\tag{61}$$

Note that $\{\varepsilon_{i1}\}$ is a m.d.s. For any fixed τ,

$$E\left|\frac{1}{n^{1-\beta}l(n)a_n}I_{1n}(T,\tau)\right|^2\le C\left(n^{1-\beta}l(n)a_n\right)^{-2}\sum_{i=\lfloor nT\rfloor+1}^{\infty}\left(\sum_{t=1}^{\lfloor n\tau\rfloor}|d_{t+i}|\right)^2E\left|\varepsilon_{11}\right|^2$$

$$\le C\left[\frac{1}{T^{2\beta-1}}\right]\left[\lfloor nT\rfloor^{2\beta-1}\sum_{i=\lfloor nT\rfloor+1}^{\infty}\left(\frac{1}{i}\right)^{2\beta}\right]\left[\frac{n}{a_n^2}E\left|\varepsilon_{11}\right|^2\right]\le C\left[\frac{1}{T^{2\beta-1}}\right]\to 0,\tag{62}$$

by letting $T\to\infty$. Similarly, for $\mu\le\tau$,

$$E\left(\frac{1}{n^{1-\beta}l(n)a_n}\left[I_{1n}(T,\tau)-I_{1n}(T,\mu)\right]\right)^2$$
$$\le C\left(n^{1-\beta}l(n)a_n\right)^{-2}\sum_{i=\lfloor nT\rfloor+1}^{\infty}\left(\sum_{t=\lfloor n\mu\rfloor+1}^{\lfloor n\tau\rfloor}|d_{t+i}|\right)^2E\varepsilon_{11}^2$$
$$\le C(\tau-\mu)^2T^{-2\beta+1}.\tag{63}$$

By Theorem 12.3 in [12], we have

$$\sup_{0\le\tau\le 1}\frac{1}{n^{1-\beta}l(n)a_n}\left|I_{1n}(T,\tau)\right|\xrightarrow{P}0,\quad\text{as }T\to\infty.\tag{64}$$

On the other hand, $\{\varepsilon_{i2}\}$ is also a m.d.s. For any $1\le 1/\beta<q<\alpha$,

$$E\left|\frac{1}{n^{1-\beta}l(n)a_n}I_{2n}(T,\tau)\right|^q\le C\left(n^{1-\beta}l(n)a_n\right)^{-q}\sum_{i=\lfloor nT\rfloor+1}^{\infty}\left(\sum_{t=1}^{\lfloor n\tau\rfloor}|d_{t+i}|\right)^qE\left|\varepsilon_{12}\right|^q$$
$$\le CT^{-\beta q+1}\to 0,$$

by letting $T\to\infty$. Similar to above, for $\mu\le\tau$,

$$E\left(\frac{1}{n^{1-\beta}l(n)a_n}\left[I_{2n}(T,\tau)-I_{2n}(T,\mu)\right]\right)^q\le C(\tau-\mu)^\gamma T^{-\beta q+1}.$$

Thus, by Theorem 12.3 in [12] again, we also have

$$\sup_{0\le\tau\le 1}\frac{1}{n^{1-\beta}l(n)a_n}\left|I_{2n}(T,\tau)\right|\xrightarrow{P}0,\quad\text{as }T\to\infty.\tag{65}$$

Combining (64) with (65) yields (60). To show (61), let $I_{3n}^+(T,\tau)=\sum_{i=-\lfloor nT\rfloor}^{\lfloor n\tau\rfloor}\sum_{t=i\vee 1}^{\lfloor n\tau\rfloor}d_{t-i}^+\varepsilon_i$ and $I_{3n}^-(T,\tau)=\sum_{i=-\lfloor nT\rfloor}^{\lfloor n\tau\rfloor}\sum_{t=i\vee 1}^{\lfloor n\tau\rfloor}d_{t-i}^-\varepsilon_i$. We first show the $f.d.d.$ convergence of $I_{3n}^+(T,\tau)$. Denote the time-space point process $N_n(ds,dr)=\sum_{t=-\lfloor nT\rfloor}^{n}\delta_{(t/n,\varepsilon_t/a_n)}(ds,dr)$. By

Theorem 2.3 in [9], for any $\varrho > 0$,

$$N_n \xrightarrow{\mathcal{L}} N, \quad \text{in } [-T, 1] \times [-\varrho, \varrho]^c. \tag{66}$$

Let $X_n^+(s,\tau) := n^{\beta-1} l^{-1}(n) \sum_{t=\lfloor ns \rfloor \vee 1}^{\lfloor n\tau \rfloor} d_{t-\lfloor ns \rfloor}^+$ and $K^+ := \lim_{n\to\infty} \frac{1}{n^{1-\beta} l(n)} \sum_{j=0}^n d_j^+$.

$$\text{Note } X_n^+(s,\tau) = \begin{cases} 0, & 0 \le \tau \le s, \\ n^{\beta-1} l^{-1}(n) \sum_{j=0}^{\lfloor n\tau \rfloor - \lfloor ns \rfloor} d_j^+, & 0 \le s \le \tau, \\ n^{\beta-1} l^{-1}(n) \left(\sum_{j=0}^{\lfloor n\tau \rfloor - \lfloor ns \rfloor} d_j^+ - \sum_{j=0}^{-\lfloor ns \rfloor} d_j^+ \right), & s \le 0 \le \tau. \end{cases} \tag{67}$$

$$\to K^+ X(s,\tau), \tag{68}$$

uniformly in $s \in [-T, 1]$, where fixing a τ, $X(.,\tau)$ is defined as in (24). For any $\epsilon < \alpha$.

$$\|X_n^+ - K^+ X\|_{\alpha,\epsilon} \to 0,$$

$$\|f\|_{\alpha,\epsilon} = \max \left\{ \int_{-\infty}^{\tau} |f(s)|^{\alpha-\epsilon} ds, \left(\int_{-\infty}^{\tau} |f(s)|^{\alpha+\epsilon} ds \right)^{\frac{\alpha-\epsilon}{\alpha+\epsilon}} \right\}. \tag{69}$$

Further, for any $\varrho > 0$,

$$\begin{aligned} \frac{I_{3n}^+(T,\tau)}{n^{1-\beta} l(n) a_n} &= \frac{1}{n^{1-\beta} l(n) a_n} \sum_{i=-\lfloor nT \rfloor}^{\lfloor n\tau \rfloor} \sum_{t=i\vee 1}^{\lfloor n\tau \rfloor} d_{t-i}^+ \varepsilon_i \\ &= \int_{-T}^{\tau+} \int_{-\infty}^{\infty} X_n^+(s,\tau) r I(|r| > \varrho) N_n(ds, dr) \\ &\quad + \int_{-T}^{\tau+} \int_{-\infty}^{\infty} X_n^+(s,\tau) r I(|r| \le \varrho) N_n(ds, dr) \\ &=: I_{3n,1}^+(T,\tau) + I_{3n,2}^+(T,\tau), \quad \text{where} \end{aligned}$$

$$\mathrm{E}\left| I_{3n,2}^+(T,\tau) \right|^2 = \sum_{i=-\lfloor nT \rfloor}^{\lfloor n\tau \rfloor} \left| \frac{\sum_{t=i\vee 1}^{\lfloor n\tau \rfloor} d_{t-i}^+}{n^{1-\beta} l(n)} \right|^2 \mathrm{E}\left| \frac{\varepsilon_i}{a_n} I\left(\left| \frac{\varepsilon_i}{a_n} \right| \le \varrho \right) \right|^2 \to 0, \quad \text{as } \varrho \to 0. \tag{70}$$

On the other hand, given (66), Lemma 4.1(a), (68) and (69), by arguments similar to Theorem 3.1 in [32] (see also Proposition 6.4 in [33], that also considered the dependent case), as $n \to \infty$,

$$I_{3n,1}^+(T,\tau) \xrightarrow{\mathcal{L}} K^+ \int_{-T}^{\tau+} \int_{[-\varrho,\varrho]^c} X(s,\tau) r N(ds, dr). \tag{71}$$

Recall the definition of $\int_{-\infty}^{\tau} X(s,\tau) d\xi_\alpha(s)$ in (25). Combining (70) with (71), we have

$$\frac{I_{3n}^+(T,\tau)}{n^{1-\beta} l(n) a_n} \xrightarrow{\mathcal{L}} K^+ \int_{-\infty}^{\tau} X(s,\tau) d\xi_\alpha(s),$$

as $\varrho \to 0$ followed by $T \to \infty$, By the standard Cramér–Wold's device, for $\tau \in [0, 1]$,

$$\frac{I_{3n}^{+}(T,\tau)}{n^{1-\beta}l(n)a_n} \xrightarrow{f.d.d.} K^{+} \int_{-\infty}^{\tau} X(s,\tau)d\xi_{\alpha}(s). \tag{72}$$

Replacing Theorem 5.1 in [32] by (72), the proof of Theorem 3.7 in [6] goes through and thus

$$\frac{I_{3n}^{+}(T,\tau)}{n^{1-\beta}l(n)a_n} \overset{M_1}{\Longrightarrow} K^{+} \int_{-\infty}^{\tau} X(s,\tau)d\xi_{\alpha}(s). \tag{73}$$

By the arguments above, we also have

$$\frac{I_{3n}^{-}(T,\tau)}{n^{1-\beta}l(n)a_n} \overset{M_1}{\Longrightarrow} K^{-} \int_{-\infty}^{\tau} X(s,\tau)d\xi_{\alpha}(s). \tag{74}$$

Further, it is not difficult to show, using the arguments for (30) in [6], that for any $0 \le \tau_1 < \cdots < \tau_k \le 1, k \ge 1$,

$$\begin{aligned} &\frac{1}{n^{1-\beta}l(n)a_n}\left(I_{3n}^{+}(T,\tau_1), I_{3n}^{-}(T,\tau_1), \dots, I_{3n}^{+}(T,\tau_k), I_{3n}^{-}(T,\tau_k)\right) \\ &\xrightarrow{\mathcal{L}} \Bigg(K^{+}\int_{-\infty}^{\tau_1} X(s,\tau)d\xi_{\alpha}(s), K^{-}\int_{-\infty}^{\tau_1} X(s,\tau)d\xi_{\alpha}(s), \dots K^{+}\int_{-\infty}^{\tau_k} X(s,\tau)d\xi_{\alpha}(s), \\ &\quad K^{-}\int_{-\infty}^{\tau_k} X(s,\tau)d\xi_{\alpha}(s)\Bigg). \end{aligned} \tag{75}$$

With (73)–(75), Theorem 2.15 in [6] implies (61). Thus (21) is proved. Re-defining $\varepsilon_{i1} := \varepsilon_i I(\sigma_i \le \sqrt{nc_0 \log n})$ and $\varepsilon_{i2} = \varepsilon_i I(\sigma_i > \sqrt{nc_0 \log n})$ in (59), Eq. (22) follows by proving:

$$\sup_{0\le\tau\le1} \frac{1}{n^{1-\beta}l(n)\sqrt{nc_0\log n}}(I_{1n}(T,\tau) + I_{2n}(T,\tau)) \xrightarrow{P} 0, \quad \text{and} \tag{76}$$

$$\frac{1}{n^{1-\beta}l(n)\sqrt{nc_0\log n}} I_{3n}(T,\tau) \overset{J_1}{\Longrightarrow} K\int_{-\infty}^{\tau} X(s,\tau)dW(s). \tag{77}$$

The proof of (76) is exactly the same as that of (60), with the above definitions of ε_{i1} and ε_{i2}, and replacing a_n by $\sqrt{nc_0 \log n}$. Let $X_n(s,\tau) := n^{\beta-1}l^{-1}(n)\sum_{t=\lfloor ns\rfloor\vee 1}^{\lfloor n\tau\rfloor} d_{t-\lfloor ns\rfloor}$. Eqs. (68)–(69) hold with $X_n^{+}(s,\tau)$ replaced by $X_n(s,\tau)$ and K^{+} replaced by K. By Lemma 4.1(b) and the same argument as in (73), it follows that (77) holds in finite-dimension distribution, by letting $n \to \infty$ and $T \to \infty$. The tightness can be proved as in Theorem 2 in [49]. Eq. (22) is thus proved. Eq. (23) follows by Theorem 2 in [49]. □

Lemma 4.2. *Suppose Assumptions 2.1–2.3 hold with α given by (4). For $h \ge 1$,*

$$\text{(a)}\ \frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor} \varepsilon_t\varepsilon_{t-h} \overset{S}{\Longrightarrow} S_{\alpha/2}^{(h)}(\tau), \quad 0 < \alpha < 4; \tag{78}$$

$$\text{(b)}\ \frac{1}{\sqrt{n\log n}}\sum_{t=1}^{\lfloor n\tau\rfloor} \varepsilon_t\varepsilon_{t-h} \overset{J_1}{\Longrightarrow} \sqrt{c_0\,(a+b)^{h-1}\left(a+bE\eta_1^4\right)}W(\tau), \quad \alpha = 4; \tag{79}$$

$$\text{(c)}\ \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t\varepsilon_{t-h}\overset{J_1}{\Longrightarrow}\sqrt{E\left(\varepsilon_1^2\varepsilon_{1-h}^2\right)}W(\tau),\quad \alpha>4; \tag{80}$$

where $S_{\alpha/2}^{(h)}(\cdot)$ is a $\alpha/2$-stable process defined as in (29). □

Proof. We first show (78) where $0<\alpha<4$. Let $S_n(\tau)=\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t\varepsilon_{t-h}/a_n^2$. By the arguments for Theorem 2.1(b) in [17] (see also Lemmas A.5–A.6 there),

$$S_n(\tau)\overset{f.d.d.}{\longrightarrow}S_{\alpha/2}^{(h)}(\tau). \tag{81}$$

Further, the arguments for relative compactness in the proof of Lemma 4.1 go through and thus (78) is proved. Next, we show (79) where $\alpha=4$. Let $k_n=\sqrt{n\log\log n}$ and $\nu_h=c_0(a+b)^{h-1}\left(a+bE\eta_1^4\right)$. Define $\zeta_{t1}=\varepsilon_t\varepsilon_{t-h}I\left(\sigma_t\left|\varepsilon_{t-h}\right|\le k_n\right)/\sqrt{n\nu_h\log n}$ and $\zeta_{t2}=\varepsilon_t\varepsilon_{t-h}I\left(\sigma_t\left|\varepsilon_{t-h}\right|>k_n\right)/\sqrt{n\nu_h\log n}$. Therefore,

$$\frac{1}{\sqrt{n\nu_h\log n}}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t\varepsilon_{t-h}=\sum_{t=1}^{\lfloor n\tau\rfloor}\zeta_{t1}+\sum_{t=1}^{\lfloor n\tau\rfloor}\zeta_{t2}.$$

For any $0<\epsilon<2$,

$$\begin{aligned}E\left[\left|\sup_{-T\le\tau\le1}\sum_{t=1}^{\lfloor n\tau\rfloor}\zeta_{t2}\right|^{2-\epsilon}\right]&\le\frac{C}{(n\log n)^{1-\epsilon/2}}\sum_{t=\lfloor -nT\rfloor}^{n}E\left|\varepsilon_t\varepsilon_{t-h}\right|^{2-\epsilon}I\left(\sigma_t\left|\varepsilon_{t-h}\right|>k_n\right)\\&\le C\frac{(n\log\log n)^{1-\epsilon/2}}{(n\log n)^{1-\epsilon/2}}\longrightarrow 0.\end{aligned} \tag{82}$$

It follows that $\sup_{-T\le\tau\le1}\sum_{t=1}^{\lfloor n\tau\rfloor}\zeta_{t2}\overset{P}{\longrightarrow}0$. Thus, it remains to show:

$$\sum_{t=1}^{\lfloor n\tau\rfloor}\zeta_{t1}\overset{J_1}{\Longrightarrow}W(\tau). \tag{83}$$

Eq. (83) follows by the proof of (6.3) in [17], if

$$\sum_{t=1}^{\lfloor n\tau\rfloor}E\left(\zeta_{t1}^2|\mathcal{F}_{t-1}\right)=\frac{E\eta_1^2}{n\nu_h\log n}\sum_{t=1}^{\lfloor n\tau\rfloor}\sigma_t^2\varepsilon_{t-h}^2I\left(\sigma_t\left|\varepsilon_{t-h}\right|\le k_n\right)\overset{P}{\longrightarrow}\tau. \tag{84}$$

See (6.5) there. Let $A_k:=a+b\eta_{k-1}^2$. Note:

$$\begin{aligned}e_t&:=\sigma_t^2\varepsilon_{t-h}^2=\prod_{k=t-h+1}^{t}A_k\sigma_{t-h}^4\eta_{t-h}^2+\omega\left[1+\sum_{k=t-h+1}^{t-1}\prod_{m=k+1}^{t}A_m\right]\varepsilon_{t-h}^2\\&=:e_{t1}+e_{t2},\quad\text{and}\\I\left(e_t\le k_n^2\right)&=I\left(e_{t1}\le k_n^2\right)-I\left(e_t>k_n^2,e_{t1}\le k_n^2\right).\end{aligned}$$

It is not difficult to see that for any $0<\varrho<1$,

$$E\left|\sup_{0\le\tau\le1}\frac{1}{n\nu_h\log n}\sum_{t=1}^{\lfloor n\tau\rfloor}e_tI\left(e_t>k_n^2,e_{t1}\le k_n^2\right)\right|^{\varrho}\le C\frac{(n\log\log n)^{\varrho}}{(n\log n)^{\varrho}}\longrightarrow 0, \tag{85}$$

$$\frac{1}{n\nu_h\log n}\sum_{t=1}^{\lfloor n\tau\rfloor}Ee_{t2}I\left(e_{t1}\le k_n^2\right)\le\frac{C}{\log n}\longrightarrow 0. \tag{86}$$

Note $\lim_{x\to\infty}xP\left(\sigma_{1-h}^4\eta_{1-h}^2\prod_{k=2-h}^1A_k>x\right)=c_0E\left(\eta_{1-h}^2\prod_{k=2-h}^1A_k\right)=\nu_h$ (by Proposition 3 in [14]), and $E\left[\sigma_{1-h}^4\eta_{1-h}^2\prod_{k=2-h}^1A_kI\left(\sigma_{1-h}^4\eta_{1-h}^2\prod_{k=2-h}^1A_k\le k_n^2\right)\right]\sim\nu_h\log n$. As a result,

$$Ee_{11}I\left(e_{11}\le k_n^2\right)\sim\nu_h\log n. \tag{87}$$

Eq. (84) follows by (85)–(87), and the arguments for (6.6) in [17]. Thus (79) is proved. Finally, when $\alpha>4$, $\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t\varepsilon_{t-h}$ is a square integrable martingale. Eq. (80) follows by Theorem 18.3 in [13]. □

Lemma 4.3. *Suppose Assumptions 2.1–2.3 hold with α given by (4).*

$$\text{(a)}\ \frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor}\left(\varepsilon_t^2-c_n\right)\overset{M_1}{\Longrightarrow}S_{\alpha/2}^{(0)}(\tau),\quad 0<\alpha<4; \tag{88}$$

$$\text{(b)}\ \frac{1}{\sqrt{n\log n}}\sum_{t=1}^{\lfloor n\tau\rfloor}\left(\varepsilon_t^2-E\sigma_1^2\right)\overset{J_1}{\Longrightarrow}\sqrt{c_0E\left(\eta_1^4-1\right)}\left(\frac{1-a}{1-a-b}\right)W(\tau),\quad\alpha=4; \tag{89}$$

$$\text{(c)}\ \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\tau\rfloor}\left(\varepsilon_t^2-E\sigma_1^2\right)\overset{J_1}{\Longrightarrow}\sqrt{E\left(\sigma_1^4\right)E\left(\eta_1^4-1\right)}\left(\frac{1-a}{1-a-b}\right)W(\tau),\quad\alpha>4; \tag{90}$$

where $c_n=0$ for $0<\alpha<2$, $c_n=c_0\log n$ for $\alpha=2$, and $c_n=E\sigma_1^2$ for $2<\alpha<4$, and $S_{\alpha/2}^{(0)}(\cdot)$ is a $\alpha/2$-stable process defined as in (29)–(31). □

Remark 4.1. As one can see in Remark 3.1, Theorem 4.3 in [9] can be used to prove Lemma 4.1(a). For $2\le\alpha<4$, if for any $\epsilon>0$,

$$\lim_{\varrho\downarrow0}\limsup_{n\uparrow\infty}P\left[\sup_{0\le\tau\le1}\left|\frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor}\left(\varepsilon_t^2I\left(|\varepsilon_t|\le\varrho a_n\right)-E\left(\varepsilon_t^2I\left(|\varepsilon_t|\le\varrho a_n\right)\right)\right)\right|>\epsilon\right]=0, \tag{91}$$

that theorem may be used to prove Lemma 4.3(a) (see their Condition 3.3 and Example 4.4). Condition (91) is not straightforward to check though. In the following, we attempt another line of proof. □

Proof of Lemma 4.3. We first consider (88)–(89). As the sample path of Brownian motion is almost surely continuous, to prove (89) under J_1, it suffices to prove it under M_1. In turn, to prove (88)–(89) under M_1, by Theorem 1 in [38], it suffices to show the following:

(a) $\{\varepsilon_t^2\}$ is an associate sequence, and
(b) the weak convergence of finite-dimension distribution (f.d.d.).

We first show (a). For any $N \geq 1$, define $\varepsilon_{t,N}^2 = \omega\left[1 + \sum_{k=t-N+1}^{t-1} \prod_{m=k+1}^{t} A_m\right]\eta_t^2$, where $A_t := a + b\eta_{t-1}^2$. As $\sigma_t^2 = \prod_{k=t-N+1}^{t} A_k \sigma_{t-N}^2 + \omega\left[1 + \sum_{k=t-N+1}^{t-1} \prod_{m=k+1}^{t} A_m\right]$, $\varepsilon_t^2 = \sigma_t^2\eta_t^2$ and $\rho(\alpha) = E\,|A_1|^{\alpha/4} < 1$ (see the argument in Lemma A.1), it follows that $\lim_{N\to\infty} \varepsilon_{t,N}^2 = \varepsilon_t^2$ a.s. Thus, by P5 in [22], it suffices to show $\{\varepsilon_{t,N}^2\}$ is an associate sequence. Write $\varepsilon_{t,N}^2 = f\left(\eta_t^2, \eta_{t-1}^2, \ldots, \eta_{t-N}^2\right)$, then the function $f(\cdot)$ is coordinate-wise non-decreasing. Since for any $N > 0$, $\left(\eta_t^2, \eta_{t-1}^2, \ldots, \eta_{t-N}^2\right)$ is an associate sequence (by Theorem 2.1 in [22]), so is $\{\varepsilon_{tN}^2\}$ (by P4 of [22]). Next we turn to (b). The f.d.d. of (88) follows by (10) and the point-process technique.[9] It remains to show the f.d.d. of (89). Note that for $\alpha > 2$, $(1-a-b)\,E\sigma_1^2 = \omega$. It follows that

$$\sigma_{t+1}^2 - E\sigma_1^2 = -(a+b)\,E\sigma_1^2 + a\sigma_t^2 + b\varepsilon_t^2 = b\sigma_t^2\left(\eta_t^2 - 1\right) + (a+b)\left(\sigma_t^2 - E\sigma_1^2\right),$$

$$\sum_{t=1}^{n}\left(\sigma_{t+1}^2 - E\sigma_1^2\right) = \sum_{t=1}^{n} b\sigma_t^2(\eta_t^2 - 1) + (a+b)\sum_{t=1}^{n}\left(\sigma_t^2 - E\sigma_1^2\right). \tag{92}$$

As a result, we have

$$(1-a-b)\sum_{t=1}^{n}\left(\sigma_t^2 - E\sigma_1^2\right) = \sum_{t=1}^{n} b\sigma_t^2\left(\eta_t^2 - 1\right) + \sigma_1^2 - \sigma_{n+1}^2, \tag{93}$$

$$\begin{aligned}\sum_{t=1}^{n}\left(\varepsilon_t^2 - E\sigma_1^2\right) &= \sum_{t=1}^{n}\sigma_t^2\left(\eta_t^2 - 1\right) + \sum_{t=1}^{n}\left(\sigma_t^2 - E\sigma_1^2\right) \\ &= \frac{1-a}{1-a-b}\sum_{t=1}^{n}\sigma_t^2\left(\eta_t^2 - 1\right) + \frac{\sigma_1^2 - \sigma_{n+1}^2}{1-a-b}.\end{aligned} \tag{94}$$

Set $k_n = \sqrt{nc_0 E\left(\eta_1^4 - 1\right)\log n}$. Since $k_n^{-1}\left(\sigma_1^2 - \sigma_{n+1}^2\right) \xrightarrow{P} 0$, by (94),

$$\frac{1}{k_n}\sum_{t=1}^{n}\left(\varepsilon_t^2 - E\sigma_1^2\right) \overset{\mathcal{L}}{=} \left(\frac{1-a}{1-a-b}\right)\frac{1}{k_n}\sum_{t=1}^{n}\sigma_t^2\left(\eta_t^2 - 1\right). \tag{95}$$

Let $\varepsilon_{t1} = \sigma_t^2(\eta_t^2 - 1)I(\sigma_t \leq (n\log\log n)^{1/4})$ and $\varepsilon_{t2} = \sigma_t^2(\eta_t^2 - 1)I(\sigma_t > (n\log\log n)^{1/4})$. By arguments for (44) in Lemma 4.1(a), it is easy to show that:

$$\frac{1}{k_n}\sum_{t=1}^{n}\sigma_t^2\left(\eta_t^2 - 1\right) \xrightarrow{\mathcal{L}} W(1). \tag{96}$$

Eqs. (95)–(96) and the Cramér–Wold device gives the f.d.d. of (89), as desired. For the proof of (90), observe that $\{\sigma_t\}$ is a β-mixing process with exponential decay (see, for instance, Theorem 3 in [23]), it follows that $\{\sigma_t^2\left(\eta_t^2 - 1\right)\}$ also satisfies the β-mixing condition with exponential decay. Eq. (90) follows by Corollary 1 in [26]. □

Proof of Theorem 3.3. We first show (32) where $0 < \alpha < 4$. Note:

$$u_t u_{t-k} - \left(\sum_{l=0}^{\infty} d_l d_{l+k}\right)c_n = \sum_{l=0}^{\infty} d_l d_{l+k}\left(\varepsilon_{t-l}^2 - c_n\right) + \sum_{h=1}^{H}\sum_{l=0}^{\infty} d_l d_{l+h+k}\varepsilon_{t-l}\varepsilon_{t-l-h}$$

[9] Alternatively, it also follows by Theorem 1.15 in [41].

$$+\sum_{h=1}^{H}\sum_{l=0\vee(k-h)}^{\infty} d_l d_{l+h-k}\varepsilon_{t-l}\varepsilon_{t-l-h} + \sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} d_l d_{l+h+k}\varepsilon_{t-l}\varepsilon_{t-l-h}$$
$$+\sum_{h=H+1}^{\infty}\sum_{l=0\vee(k-h)}^{\infty} d_l d_{l+h-k}\varepsilon_{t-l}\varepsilon_{t-l-h}. \tag{97}$$

As the proofs are similar, we only give that for $k=0$, consider:

$$\sum_{l=0}^{\infty} d_l^2\left(\varepsilon_{t-l}^2-c_n\right)+2\sum_{h=1}^{H}\sum_{l=0}^{\infty} d_l d_{l+h}\varepsilon_{t-l}\varepsilon_{t-l-h}+2\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} d_l d_{l+h}\varepsilon_{t-l}\varepsilon_{t-l-h}$$
$$=: I_{1t}+I_{2t}(H)+I_{3t}(H). \tag{98}$$

For any positive integers l and H, we define an $(H+1)$-dimensional random vector:

$$\begin{aligned} \boldsymbol{Z}_{t,l,H} &= \left(\varepsilon_{t-l}^2-c_n, \varepsilon_{t-l}\varepsilon_{t-l-1},\ldots,\varepsilon_{t-l}\varepsilon_{t-l-H}\right)\\ &=: \left(Z_{t,l}^{(0)}, Z_{t,l}^{(1)},\ldots,Z_{t,l}^{(H)}\right), \quad t\in\mathcal{Z}. \end{aligned} \tag{99}$$

Since convergence in M_1 implies convergence in S, by Lemmas 4.2(a), 4.3(a) and (10), for any positive integer l,

$$\frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor}\boldsymbol{Z}_{t,l,H} \stackrel{S}{\Longrightarrow} \left(S_{\alpha/2}^{(0)}(\tau), S_{\alpha/2}^{(1)}(\tau),\ldots,S_{\alpha/2}^{(H)}(\tau)\right) =: \boldsymbol{S}_H(\tau).$$

Thus for any positive integers L and H,

$$\frac{1}{a_n^2}\left(\sum_{t=1}^{\lfloor n\tau\rfloor}\boldsymbol{Z}_{t,0,H}, \sum_{t=1}^{\lfloor n\tau\rfloor}\boldsymbol{Z}_{t,1,H},\ldots,\sum_{t=1}^{\lfloor n\tau\rfloor}\boldsymbol{Z}_{t,L,H}\right) \stackrel{S}{\Longrightarrow} \left(\boldsymbol{S}_H(\tau),\boldsymbol{S}_H(\tau),\ldots,\boldsymbol{S}_H(\tau)\right)_{1\times L}. \tag{100}$$

By (100) and the fact that addition is sequentially continuous with respect to S topology (see [31]), it follows that

$$\begin{aligned} \frac{1}{a_n^2}S_{n,L,H}(\tau) &:= \frac{1}{a_n^2}\Big(\sum_{l=0}^{L} d_l^2\sum_{t=1}^{\lfloor n\tau\rfloor}\left(\varepsilon_{t-l}^2-c_n\right)+2\sum_{l=0}^{L} d_l d_{l+1}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-l}\varepsilon_{t-l-1}\\ &\qquad +\cdots+2\sum_{l=0}^{L} d_l d_{l+H}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-l}\varepsilon_{t-l-H}\Big)\\ &\stackrel{S}{\Longrightarrow} \sum_{l=0}^{L} d_l^2 S_{\alpha/2}^{(0)}(\tau)+2\sum_{l=0}^{L} d_l d_{l+1}S_{\alpha/2}^{(1)}(\tau)+\cdots+2\sum_{l=0}^{L} d_l d_{l+H}S_{\alpha/2}^{(H)}(\tau)\\ &=: S_{L,H}(\tau). \end{aligned} \tag{101}$$

We first consider the sub-case $0<\alpha<2$. Fix an H and for any L,

$$\sup_{0\le\tau\le1}\frac{1}{a_n^2}\left|\sum_{t=1}^{\lfloor n\tau\rfloor}(I_{1t}+I_{2t}(H))-S_{n,L,H}(\tau)\right| \le \frac{1}{a_n^2}\sum_{t=1}^{n}\sum_{l=L+1}^{\infty}\Big[d_l^2\varepsilon_{t-l}^2 I\left(\varepsilon_{t-l}^2\le a_n^2\right)$$

$$
\begin{aligned}
&\quad + \sum_{h=1}^{H} |d_l d_{l+h} \varepsilon_{t-l} \varepsilon_{t-l-h}| \, I\left(|\varepsilon_{t-l}\varepsilon_{t-l-h}| \le a_n^2\right)\Bigg] \\
&\quad + \frac{1}{a_n^2} \sum_{t=1}^{n} \sum_{l=L+1}^{\infty} \Bigg[d_l^2 \varepsilon_{t-l}^2 I\left(\varepsilon_{t-l}^2 > a_n^2\right) \\
&\quad + \sum_{h=1}^{H} |d_l d_{l+h} \varepsilon_{t-l} \varepsilon_{t-l-h}| \, I\left(|\varepsilon_{t-l}\varepsilon_{t-l-h}| > a_n^2\right)\Bigg] \\
&=: \Pi_1(H, L) + \Pi_2(H, L).
\end{aligned}
$$

By (27) it is easy to show that, uniformly in H, $\lim_{L\to\infty} \lim_{n\to\infty} E|\Pi_1(H, L)| = 0$ and $\lim_{L\to\infty} \lim_{n\to\infty} E|\Pi_2(H, L)|^{p/2} = 0$. Thus for any $\epsilon > 0$ and uniformly in H,

$$
\lim_{L\to\infty} \lim_{n\to\infty} P\left(\sup_{0\le\tau\le1} \frac{1}{a_n^2} \left| \sum_{t=1}^{\lfloor n\tau \rfloor} (I_{1t} + I_{2t}(H)) - S_{n,L,H}(\tau) \right| > \epsilon \right) = 0. \tag{102}
$$

For the sub-case $\alpha = 2$, fix an H and for any L,

$$
\begin{aligned}
\sum_{t=1}^{\lfloor n\tau \rfloor} (I_{1t} + I_{2t}(H)) - S_{n,L,H}(\tau) &= \sum_{j=-\infty}^{-L} \left(\sum_{t=1}^{\lfloor n\tau \rfloor} d_{t-j}^2 \right) \left(\varepsilon_j^2 - c_n\right) \\
&\quad + \sum_{t=1}^{\lfloor n\tau \rfloor} \sum_{l=L+1}^{\infty} \sum_{h=1}^{H} d_l d_{l+h} \varepsilon_{t-l} \varepsilon_{t-l-h} + \sum_{j=1-L}^{\lfloor n\tau \rfloor - L} \left(\sum_{t=L+j}^{\lfloor n\tau \rfloor} d_{t-j}^2 \right) (\varepsilon_j^2 - c_n) \\
&=: \Lambda_{1n}(L, \tau) + \Lambda_{2n}(H, L, \tau) + \Lambda_{3n}(L, \tau).
\end{aligned} \tag{103}
$$

For p given in (27), $E|\varepsilon_1|^p < \infty$. It follows that as $L \to \infty$,

$$
E \left| \sup_{0\le\tau\le1} n^{-1} \Lambda_{1n}(L, \tau) \right|^{p/2} \le \left(\sum_{j=-\infty}^{-L} \left|d_{1-j}\right|^p \right) E\, |\varepsilon_1|^p \to 0. \tag{104}
$$

For $\Lambda_{2n}(H, L, \tau)$, let $\zeta_{t1}(h) = \varepsilon_t \varepsilon_{t-h} I\,(|\varepsilon_t \varepsilon_{t-h}| \le n)$ and $\zeta_{t2}(h) = \varepsilon_t \varepsilon_{t-h} I\,(|\varepsilon_t \varepsilon_{t-h}| > n)$. As in the proof of Theorem 3.1, applying Doob's inequality to $\sum_{t=1}^{\lfloor n\tau \rfloor} \sum_{l=L+1}^{\infty} \sum_{h=1}^{H} d_l d_{l+h} \zeta_{t1}(h)$ and Karamata's theorem to $\sum_{t=1}^{[n\tau]} \sum_{l=L+1}^{\infty} \sum_{h=1}^{H} d_l d_{l+h} \zeta_{t2}(h)$, it follows that as $L \to \infty$,

$$
\sup_{0\le\tau\le1} n^{-1} \Lambda_{2n}(H, L, \tau) \xrightarrow{P} 0, \tag{105}
$$

uniformly in H. By Lemma A.3, $\sup_{0\le\tau\le1} n^{-1} \Lambda_{3n}(L, \tau) \xrightarrow{P} 0$ as $L \to \infty$. Together with (104)–(105), Eq. (102) holds for $\alpha = 2$. Next we consider the sub-case where $2 < \alpha < 4$. By the representation in (95) and the fact that for any $h \ge 1$, $\{\varepsilon_t \varepsilon_{t-h}\}$ is a m.d.s., along the line in the proof of Theorem 3.1, we can show that (102) also holds. All in all, for $0 < \alpha < 4$, by (101) and (102),

$$
\frac{1}{a_n^2} \sum_{t=1}^{\lfloor n\tau \rfloor} (I_{1t} + I_{2t}(H)) \overset{S}{\Longrightarrow} \lim_{L\to\infty} S_{L,H}(\tau) =: S_H(\tau). \tag{106}
$$

But by Lemma A.1,

$$\sup_{0\le\tau\le1}\left|\frac{1}{a_n^2}\sum_{t=1}^{\lfloor n\tau\rfloor}\left(u_t^2-\sum_{l=0}^{\infty}d_l^2c_n\right)-\frac{1}{a_n^2}\sum_{t=1}^{n}(I_{1t}+I_{2t}(H))\right|\xrightarrow{P}0, \tag{107}$$

by letting $n\to\infty$ and then $H\to\infty$. Thus, it remains to show

$$\lim_{H\to\infty}S_H(1)=\left(\sum_{l=0}^{\infty}d_l^2\right)\sum_{i,j\ge1}P_i^2Q_{ij}^{(0)2}+2\sum_{h=1}^{\infty}\left(\sum_{l=0}^{\infty}d_ld_{l+h}\right)\sum_{i,j\ge1}P_i^2Q_{ij}^{(0)}Q_{ij}^{(h)} \tag{108}$$

is well defined, where P_i and $Q_{ij}^{(h)}$'s are defined as in Section 2. By (78), (A.4) and (A.5), for large x, $P\left(|S_{\alpha/2}^h(1)|>x\right)\sim C\rho^hx^{-\alpha/2}$. Let p be as in (27), for any $\epsilon>0$,

$$\begin{aligned}&P\left(\left|\sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty}d_ld_{l+h}\right)\sum_{i,j\ge1}P_i^2Q_{ij}^{(0)}Q_{ij}^{(h)}\right|>\epsilon\right)\\&\le\epsilon^{-p/2}\sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty}|d_ld_{l+h}|^{p/2}\right)E\left|\sum_{i,j\ge1}P_i^2Q_{ij}^{(0)}Q_{ij}^{(h)}\right|^{p/2}\\&\le C\epsilon^{-p/2}\left(\sum_{l=0}^{\infty}|d_l|^p\right)\sum_{h=H+1}^{\infty}\rho^h\to0,\end{aligned}$$

as $H\to\infty$. By Property 1.2.17 in [45], $\lim_{H\to\infty}S_H(1)$ is well defined. When $\alpha\ge4$, by the representation in (95) and the fact that for any $h\ge1$, $\{\varepsilon_t\varepsilon_{t-h}\}$ is a m.d.s., along the line in the proof of showing (14) and (15), it is not difficult to show (33) and (34). This completes the proof. □

Proof of Theorem 3.4. The proof of (a) is the same as that of Theorem 3.3(a), with Lemma A.1 replaced by Lemma A.2. Next we turn to (b). As the proofs are similar, again we only give that for $k=0$. Recall the definition of k_n in Lemma A.1. Refer to (98). Given Lemma 4.3 and the fact that $\sum_{l=1}^{\infty}d_l^2<\infty$, it follows that for $2<\alpha\le4$ with $\beta<1-1/\alpha$, or $\alpha>4$ with $\beta<3/4$, $\sum_{t=1}^{n}I_{1t}=O_P(k_n)=o_P\left(n^{2(1-\beta)}l^2(n)\right)$. Thus, it remains to prove:

$$\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=1}^{\infty}\sum_{l=0}^{\infty}d_ld_{l+h}\varepsilon_{t-l}\varepsilon_{t-l-h}\xRightarrow{J_1}\left(E\sigma_1^2\right)U_\beta(\tau). \tag{109}$$

Careful inspection of the arguments for (A.7) shows:

$$\sup_{0\le\tau\le1}\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=1}^{\infty}\sum_{l=0}^{\infty}d_ld_{l+h}\varepsilon_{t-l-h}\left(\sigma_{t-l}-\Delta_2^{1/2}(t-l,h)\right)\eta_{t-l}=O_P(1), \tag{110}$$

where $\Delta_2(t-l,h)$ is defined as in Lemma A.1. Thus, it remains to prove:

$$\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=1}^{\infty}\sum_{l=0}^{\infty}d_ld_{l+h}\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}\xRightarrow{J_1}\left(E\sigma_1^2\right)U_\beta(\tau). \tag{111}$$

For any $0 \le s < \tau \le 1$, by arguments similar to those for (A.13)–(A.15) (with $\beta < 3/4$),

$$E\left|\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=\lfloor ns\rfloor+1}^{\lfloor n\tau\rfloor}\sum_{h=1}^{\infty}\sum_{l=0}^{\infty}d_l d_{l+h}\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}\right|^2 \le C(\tau-s)^{4(1-\beta)}, \tag{112}$$

where $4(1-\beta) > 1$. Thus, it remains to show (111) holds for $\tau = 1$. For any $0 < H < \infty$,

$$\begin{aligned}
&E\left|\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=1}^{n}\sum_{h=1}^{H}\sum_{l=0}^{\infty}d_l d_{l+h}\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}\right|^2 \\
&\le \frac{C}{n^{4(1-\beta)}l^4(n)}\sum_{j=-\infty}^{n}E\left|\sum_{h=1}^{H}\left(\sum_{t=1\vee j}^{n}d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\Delta_2^{1/2}(j,h)\eta_j\right|^2 \\
&\le \frac{C}{n^{4(1-\beta)}}\left[\sum_{t=1}^{n}\sum_{t'=1}^{n}\sum_{j=0}^{\infty}\sum_{h=1}^{H}(t+j)^{-\beta}(t'+j)^{-\beta}(t+j+h)^{-\beta}(t'+j+h)^{-\beta}\right. \\
&\quad\left.\times\sum_{j=1}^{n}\sum_{h=1}^{H}\sum_{t=j+1}^{n}\sum_{t'=j+1}^{n}(t-j)^{-\beta}(t'-j)^{-\beta}(t-j+h)^{-\beta}(t'-j+h)^{-\beta}\right] \\
&\le C\left(Hn^{-2(1-\beta)} + Hn^{1-4(1-\beta)}\right) \to 0, \quad \text{as}\, n \to \infty,
\end{aligned} \tag{113}$$

since $\beta < 3/4$. Thus, it remains to prove:

$$\frac{1}{n^{2(1-\beta)}l^2(n)}\sum_{t=1}^{n}\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty}d_l d_{l+h}\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l} \xrightarrow{\mathcal{L}} \left(E\sigma_1^2\right)U_\beta(1), \tag{114}$$

as $n \to \infty$ and $H \to \infty$. Note that for any given l and h, $\left\{\Delta_2^{1/2}(t-l,h)\eta_{t-l}, t = 1,\dots,n\right\}$ is a m.d.s. with $(\alpha+2)/2$th moment, with $\lim_{h\to\infty} E[\Delta_2(1,h)] = E\sigma_1^2$. Thus,

$$\frac{1}{\sqrt{nE\sigma_1^2}}\sum_{t=1}^{\lfloor n\tau\rfloor}\Delta_2^{1/2}(t-l,h)\eta_{t-l} \overset{J_1}{\Longrightarrow} W(\tau), \tag{115}$$

by letting $h \to \infty$. Further, by Lemma 4.1(c), we also have

$$\frac{1}{\sqrt{nE\sigma_1^2}}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_t \overset{J_1}{\Longrightarrow} W(\tau). \tag{116}$$

Using (115) and (116), as $n \to \infty$ and then $H \to \infty$, (114) can be shown along the lines in Theorem 6.1 in [24] (see also Theorems 3.1(b) and 3.3(b) in [27]). Finally we consider (c) and (d). The proofs are the same as those of Theorem 3.3(b) and (c) respectively, with Lemma A.1 replaced by Lemma A.2. □

5. Statistical applications

The results in the last section have a lot of potential applications. Zhang and Ling [51] applied Theorem 3.3 in this paper to a short-memory $AR(p)$ model:

$$Y_t = \sum_{i=1}^{p} \phi_{0i} Y_{t-i} + \varepsilon_t, \tag{117}$$

where ε_t satisfies a general version of (3), namely a power GARCH(1,1) model. The LSE of $\boldsymbol{\phi}_0 = \left(\phi_{01}, \ldots, \phi_{0p}\right)'$ is defined by

$$\hat{\boldsymbol{\phi}}_n = \left(\sum_{t=p+1}^{n} \boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'\right)^{-1} \left(\sum_{t=p+1}^{n} \boldsymbol{Y}_{t-1} Y_t\right), \tag{118}$$

where $\boldsymbol{Y}_t = \left(Y_t, Y_{t-1}, \ldots, Y_{t-p+1}\right)'$. They showed in their Theorem 1 that:

(a) $\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_0 \xrightarrow{\mathcal{L}} \Sigma_{\alpha/2}^{-1}\mathbf{Z}_{\alpha/2}, \quad 0 < \alpha < 2;$

(b) $\log n \left(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_0\right) \xrightarrow{\mathcal{L}} A_b^{-1}\mathbf{Z}_{\alpha/2}, \quad \alpha = 2;$

(c) $n^{1-2/\alpha} \left(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_0\right) \xrightarrow{\mathcal{L}} A_c^{-1}\mathbf{Z}_{\alpha/2}, \quad 2 < \alpha < 4;$

(d) $(n/\log n)^{1/2} \left(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_0\right) \xrightarrow{\mathcal{L}} A_d^{-1}\boldsymbol{N}\left(0, I_p\right), \quad \alpha = 4,$

where $\mathbf{Z}_{\alpha/2}$ is a p-dimensional stable vector with index $\alpha/2$, $\Sigma_{\alpha/2}$ is a $p \times p$ matrix whose elements are composed of stable variables with index $\alpha/2$; A_b, A_c and A_d are non-random $p \times p$ matrices. That is, the LSE is not consistent when $0 < \alpha < 2$; and it is $n^{1-2/\alpha}$-consistent when $2 < \alpha < 4$, $\log n$-consistent when $\alpha = 2$, and $n^{1/2}/\log n$-consistent when $\alpha = 4$. Furthermore, the limit distribution of the LSE is a functional of stable processes when $\alpha < 4$ which is substantially different from those with i.i.d. noises. On the other hand, Zhang, Sin and Ling [52] considered the following *unit root* process:

$$y_t = \mu + \phi y_{t-1} + u_t, \tag{119}$$

where $\mu = 0$, $\phi = 1$ and u_t is defined as in (2). The LSE of the ϕ are

$$\hat{\phi}_n = \left(\sum_{t=1}^{n} y_{t-1}^2\right)^{-1} \left(\sum_{t=1}^{n} y_{t-1} y_t\right),$$

$$\hat{\phi}_{\mu n} = \left(\sum_{t=1}^{n} (y_{t-1} - \bar{y})^2\right)^{-1} \left(\sum_{t=1}^{n} (y_{t-1} - \bar{y}) y_t\right),$$

when $\mu = 0$ is known and unknown, respectively. Using Theorems 3.1 and 3.3 in this paper (short-memory u_t's), Theorem 2.1 in Zhang, Sin and Ling [52] showed

(a) If $0 < \alpha < 2$, then

$$n\left(\hat{\phi}_n - 1\right) \xrightarrow{\mathcal{L}} \frac{\int_0^1 \xi_\alpha^-(\tau) d\xi_\alpha(\tau) + \Xi_{\alpha/2}(1)}{\int_0^1 \xi_\alpha^2(\tau) d\tau},$$

$$n\left(\hat{\phi}_{\mu n}-1\right) \xrightarrow{\mathcal{L}} \frac{\int_0^1 \xi_\alpha^-(\tau)d\xi_\alpha(\tau)-\xi_\alpha(1)\int_0^1 \xi_\alpha(\tau)d\tau+\Xi_{\alpha/2}(1)}{\int_0^1 \xi_\alpha^2(\tau)d\tau-\left(\int_0^1 \xi_\alpha(\tau)d\tau\right)^2};$$

(b) If $\alpha \geq 2$, then

$$n\left(\hat{\phi}_n-1\right) \xrightarrow{\mathcal{L}} \frac{\int_0^1 W(\tau)dW(\tau)+\frac{1}{2}\left(1-\frac{\sigma^2}{\sigma_u^2}\right)}{\int_0^1 W^2(\tau)d\tau},$$

$$n\left(\hat{\phi}_{\mu n}-1\right) \xrightarrow{\mathcal{L}} \frac{\int_0^1 W(\tau)dW(\tau)-W(1)\int_0^1 W(\tau)d\tau+\frac{1}{2}\left(1-\frac{\sigma^2}{\sigma_u^2}\right)}{\int_0^1 W^2(\tau)d\tau-\left(\int_0^1 W(\tau)d\tau\right)^2},$$

where $\xi_\alpha(\tau)$, $\Xi_{\alpha/2}(\tau)$ are two stable vectors with index α and $\alpha/2$ respectively. Using Theorems 3.2 and 3.4 in this paper (long-memory u_t's), their Theorem 2.2 shows

(a) If $1 < \alpha < 2$, then

$$n\left(\hat{\phi}_n-1\right) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2}\left(\int_{-\infty}^1 X(s,1)d\xi_\alpha(s)\right)^2}{\int_0^1 \left(\int_{-\infty}^\tau X(s,\tau)d\xi_\alpha(s)\right)^2 d\tau},$$

$$n\left(\hat{\phi}_{\mu n}-1\right) \xrightarrow{\mathcal{L}}$$
$$\frac{\frac{1}{2}\left(\int_{-\infty}^1 X(s,1)d\xi_\alpha(s)\right)^2-\left(\int_{-\infty}^1 X(s,1)d\xi_\alpha(s)\right)\left(\int_0^1\int_{-\infty}^\tau X(s,\tau)d\xi_\alpha(s)d\tau\right)}{\int_0^1 \left(\int_{-\infty}^\tau X(s,\tau)d\xi_\alpha(s)\right)^2 d\tau-\left(\int_0^1\int_{-\infty}^\tau X(s,\tau)d\xi_\alpha(s)d\tau\right)^2};$$

(b) If $\alpha \geq 2$, then

$$n\left(\hat{\phi}_n-1\right) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2}\left(\int_{-\infty}^1 X(s,1)dW(s)\right)^2}{\int_0^1 \left(\int_{-\infty}^\tau X(s,\tau)dW(s)\right)^2 d\tau},$$

$$n\left(\hat{\phi}_{\mu n}-1\right) \xrightarrow{\mathcal{L}}$$
$$\frac{\frac{1}{2}\left(\int_{-\infty}^1 X(s,1)dW(s)\right)^2-\left(\int_{-\infty}^1 X(s,1)dW(s)\right)\left(\int_0^1\int_{-\infty}^\tau X(s,\tau)dW(s)d\tau\right)}{\int_0^1 \left(\int_{-\infty}^\tau X(s,\tau)dW(s)\right)^2 d\tau-\left(\int_0^1\int_{-\infty}^\tau X(s,\tau)dW(s)d\tau\right)^2}.$$

These distributions are somewhat different from those in [44] where the noises are i.i.d. Finally, our results in the last section can be used in other applications such as testing for a change point in mean [4], or in covariance [2], and inference in mean [30] or in autocovariances and autocorrelations [39].

Appendix

Lemma A.1. *Suppose the conditions in Theorem 3.3 hold with $\alpha > 0$. Then*

$$\lim_{H\to\infty}\lim_{n\to\infty}\left|\frac{1}{k_n}\sup_{0\le\tau\le1}\sum_{t=1}^{\lfloor n\tau\rfloor} I_{3t}(H)\right| \xrightarrow{P} 0,$$
$$I_{3t}(H) := \sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} d_l d_{l+h}\varepsilon_{t-l}\varepsilon_{t-l-h}, \tag{A.1}$$

where $k_n = a_n^2$ *for* $0 < \alpha \le 4$, $k_n = \sqrt{n\log n}$ *for* $\alpha = 4$, *and* $k_n = \sqrt{n}$ *for* $\alpha > 4$. □

Proof. Denote $A_k = a + b\eta_{k-1}^2$. For all t and $h \ge 1$,

$$\sigma_t^2 = \prod_{k=t-h+1}^{t} A_k\sigma_{t-h}^2 + \omega\left[1 + \sum_{k=t-h+1}^{t-1}\prod_{m=k+1}^{t} A_m\right] =: \Delta_1(t,h) + \Delta_2(t,h).$$

$$\text{Write } I_2(\lfloor n\tau\rfloor, H) = \sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} d_l d_{l+h}\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}. \tag{A.2}$$

It follows that

$$\begin{aligned}
&\left|\sum_{t=1}^{\lfloor n\tau\rfloor} I_{3t}(H) - I_2(\lfloor n\tau\rfloor, H)\right| \\
&= \left|\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} d_l d_{l+h}\varepsilon_{t-l-h}\left(\varepsilon_{t-l} - \Delta_2^{1/2}(t-l,h)\eta_{t-l}\right)\right| \\
&= \left|\sum_{j=-\infty}^{\lfloor n\tau\rfloor}\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{\lfloor n\tau\rfloor} d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\left(\sigma_j - \Delta_2^{1/2}(j,h)\right)\eta_j\right| \\
&=: \left|I_1(\lfloor n\tau\rfloor, H)\right|.
\end{aligned} \tag{A.3}$$

By the Cauchy–Schwarz inequality and the non-degeneracy of $\{\eta_t\}$, $0 < \rho = \rho(\alpha) = EA_1^{\alpha/4} \le [EA_1^{\alpha/2}]^{1/2} = 1$. However, by Theorem 4 in [34], $\alpha/2$ is the unique solution to the equation $EA_1^x = 1$, $x > 0$. Thus, $0 < \rho < 1$. Since $\lim_{x\to\infty} x^{\alpha/2}P\left(\sigma_1^2 > x\right) = c_0$, it follows from Proposition 3 in [14] that

$$\begin{aligned}
&\lim_{y\to\infty} P\left\{\prod_{k=t-h+1}^{t} A_k^{1/2}|\eta_{t-h}\eta_t|\sigma_{t-h}^2 > y\right\} \\
&= c_0 E\left[\eta_1^2 A_1\right]^{\alpha/4} E\left|\eta_1\right|^{\alpha/2}\left[EA_1^{\alpha/4}\right]^{h-1} y^{-\alpha/2} \\
&=: k_1\rho^h y^{-\alpha/2}, \quad \text{and}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
&\lim_{y\to\infty} P\left\{\omega\left[1 + \sum_{k=t-h+1}^{t}\prod_{m=k+1}^{t} A_m\right]\eta_{t-h}^2\eta_t^2\sigma_{t-h}^2 > y\right\} \\
&= c_0 E\left\{\omega\left[1 + \sum_{k=t-h+1}^{t}\prod_{m=k+1}^{t} A_m\right]\eta_{t-h}^2\eta_t^2\right\}^{\alpha/2} y^{-\alpha/2} =: k_2 y^{-\alpha/2}.
\end{aligned} \tag{A.5}$$

We first consider $0 < \alpha \le 2$. By (A.4), it follows that for $p < \alpha$ given in (27),

$$
\begin{aligned}
&E\left|\sup_{0\le\tau\le1}\frac{1}{k_n}I_1(\lfloor n\tau\rfloor, H)\right|^{p/2}\\
&q \le \frac{C}{n^{p/\alpha}}\sum_{j=-\infty}^{n}\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{n}|d_{t-j}d_{t-j+h}|\right)^{p/2} E\left(|\varepsilon_{j-h}|^{p/2}\Delta_1^{p/4}(j,h)\right)E\left|\eta_j\right|^{p/2}\\
&\le \frac{C}{n^{p/\alpha}}\sum_{j=-\infty}^{n}\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{n}|d_{t-j}d_{t-j+h}|\right)^{p/2}\left(\rho^{p/\alpha}\right)^{h}\\
&\le Cn^{p(1/2-1/\alpha)}\left(\rho^{p/\alpha}\right)^{H+1}\to 0,
\end{aligned}
\tag{A.6}
$$

as $n\to\infty$ and $H\to\infty$. For $2 < \alpha \le 4$, write

$$
\begin{aligned}
\frac{1}{k_n}I_1(\lfloor n\tau\rfloor, H) &= \frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty}d_l d_{l+h}\varepsilon_{t-l-h}I\left(\sigma_{t-l-h}\le a_n\right)\\
&\quad\times\left(\sigma_{t-l}-\Delta_2^{1/2}(t-l,h)\right)\eta_{t-l}+\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty}d_l d_{l+h}\varepsilon_{t-l-h}I\left(\sigma_{t-l-h}> a_n\right)\\
&\quad\times\left(\sigma_{t-l}-\Delta_2^{1/2}(t-l,h)\right)\eta_{t-l} =: I_{11}(\lfloor n\tau\rfloor, H)+I_{12}(\lfloor n\tau\rfloor, H).
\end{aligned}
$$

By Doob's inequality and Karamata's theorem,

$$
\begin{aligned}
&\left[E\left|\sup_{0\le\tau\le1}I_{11}(\lfloor n\tau\rfloor, H)\right|^2\right]^{1/2} \le C\sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty}|d_l d_{l+h}|\right)\\
&\quad\times\left[E\left[\sup_{0\le\tau\le1}\frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor}\varepsilon_{t-l-h}I\left(\sigma_{t-l-h}\le a_n\right)\left(\sigma_{t-l}-\Delta_2^{1/2}(t-l,h)\right)\eta_{t-l}\right]^2\right]^{1/2}\\
&\le C\sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty}d_l^2\right)\left[\frac{1}{k_n^2}\sum_{t=1}^{n}E\left[\sigma_{t-l-h}^4 I\left(\sigma_{t-l-h}\le a_n\right)\right][EA_1]^h\right]^{1/2}\\
&\le C\sum_{h=H+1}^{\infty}[EA_1]^{h/2}\to 0,
\end{aligned}
$$

as $n\to\infty$ and $H\to\infty$. For $p < 2 < \alpha$ as that given in (27), by Karamata's theorem,

$$
\begin{aligned}
&E\left|\sup_{0\le\tau\le1}I_{12}(\lfloor n\tau\rfloor, H)\right|^{p/2}\\
&\le C\sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty}|d_l d_{l+h}|^{p/2}\right)\frac{1}{n^{p/\alpha}}\sum_{t=1}^{n}E\left[\sigma_{t-l-h}^{p}I\left(\sigma_{t-l-h}> a_n\right)\right][EA_1^{p/4}]^h\\
&\le C\sum_{h=H+1}^{\infty}[EA_1^{p/4}]^h\left(\sum_{l=0}^{\infty}|d_l|^p\right)\to 0,
\end{aligned}
$$

as $n \to \infty$ and $H \to \infty$. For $\alpha > 4$, similar to above,

$$\left[E\left|\sup_{0\le\tau\le1} I_1(\lfloor n\tau\rfloor, H)\right|^2\right]^{1/2} \le C \sum_{h=H+1}^{\infty}\left(\sum_{l=0}^{\infty} d_l^2\right)\left[\frac{1}{n}\sum_{t=1}^{n} E\left[\sigma_{t-l-h}^4\right][EA_1]^h\right]^{1/2} \to 0,$$

as $n \to \infty$ and $H \to \infty$. All in all, for $\alpha > 0$,

$$\sup_{0\le\tau\le1} \frac{1}{k_n} I_1(\lfloor n\tau\rfloor, H) \xrightarrow{P} 0. \quad \text{(A.7)}$$

Thus, it remains to show

$$\sup_{0\le\tau\le1} \frac{1}{k_n} I_2(\lfloor n\tau\rfloor, H) \xrightarrow{P} 0. \quad \text{(A.8)}$$

Again, first consider $0 < \alpha < 2$. Without loss of generality, we assume the γ given by $\sum_{l=1}^{n} |d_l|^\gamma < \infty$ satisfies $\gamma > \alpha/2$ when $\alpha < 1$. By Karamata's theorem,

$$E\left|\sup_{0\le\tau\le1} \frac{1}{k_n} I_2(\lfloor n\tau\rfloor, H)\right|^\gamma \le \frac{C}{n^{2\gamma/\alpha}} \sum_{h=H+1}^{\infty}\sum_{l=0}^{\infty} |d_l d_{l+h}|^\gamma \times \sum_{t=1}^{n} E\left|\varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}\right|^\gamma \to 0, \quad \text{(A.9)}$$

as $n \to \infty$ and $H \to \infty$. When $\alpha > 2$, note $E\varepsilon_1^2 < \infty$. By Doob's inequality,

$$\begin{aligned}\left[E\left|\sup_{0\le\tau\le1} \frac{1}{k_n} I_2(\lfloor n\tau\rfloor, H)\right|\right]^{1/2} &\le C \sum_{h=H+1}^{\infty} |d_h| \sum_{l=0}^{\infty} |d_l| \left[E\left|\sup_{0\le\tau\le1} \frac{1}{k_n}\sum_{t=1}^{\lfloor n\tau\rfloor} \varepsilon_{t-l-h}\Delta_2^{1/2}(t-l,h)\eta_{t-l}\right|^2\right]^{1/2} \\ &\le C \sum_{h=H+1}^{\infty} |d_h| \to 0, \quad \text{(A.10)}\end{aligned}$$

as $n \to \infty$ and $H \to \infty$. (A.8) follows by (A.9) and (A.10). This completes the proof. □

Lemma A.2. *Suppose the conditions in Theorem 3.4 hold with either* (a) $1 < \alpha \le 2$*; or* (b) $2 < \alpha \le 4$ *with* $\beta > 1 - 1/\alpha$*; or* (c) $\alpha > 4$ *and* $\beta > 3/4$*. Then (A.1) holds.* □

Proof. For $\alpha > 1$, careful inspection of the proof of Lemma A.1 will show that under the conditions in Lemma A.2, which considers the long-memory cases, the arguments up to (A.7) are valid. It remains to show (A.8). First consider (a). Since $\{\sum_{h=H}^{\infty}\left(\sum_{t=1\vee j}^{n} d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\Delta_2^{1/2}(t,h)\eta_j\} =: \{\xi_j\}$ and $\{\Delta_2^{1/2}(t,h)\varepsilon_{t-h}\} =: \{X_h\}$ are m.d.s. w.r.t. the σ-field $\mathcal{F}_t = \sigma\{\eta_s, s \le t\}$, by the martingale inequality (see, for instance, Lemma 1 in [50]) that for fixed τ and any $1/\beta < q < \alpha \le 2$,

$$E\left|\frac{1}{k_n} I_2(\lfloor n\tau \rfloor, H)\right|^q \le \frac{2^{q+1}}{n^{2q/\alpha}} \sum_{j=-\infty}^{n} E\left|\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{[n\tau]} d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\Delta_2^{1/2}(j,h)\eta_j\right|^q$$
$$\le \frac{2^{q+2}}{n^{2q/\alpha}} \sum_{j=-\infty}^{n}\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{n} |d_{t-j}d_{t-j+h}|\right)^q \left(E\left|\varepsilon_{j-h}\Delta_2^{1/2}(j,h)\right|^q\right) E\left|\eta_j\right|^q$$
$$\le C n^{q(1-2/\alpha)} \sum_{j=-\infty, j\neq 1}^{n} \left((1-j)^+\right)^{-q\beta} \sum_{h=H}^{\infty} h^{-q\beta} \to 0, \tag{A.11}$$

as $n \to \infty$ and $H \to \infty$. Similarly, we can show that for any $0 \le s < \tau \le 1$,

$$E\left|\frac{1}{k_n}[I_2(\lfloor n\tau \rfloor, H) - I_2(\lfloor ns \rfloor, H)]\right|^q$$
$$\le \frac{2^{q+1}}{n^{2q/\alpha}} \sum_{j=-\infty}^{n} E\left|\sum_{h=H+1}^{\infty}\left(\sum_{t=(\lfloor ns \rfloor+1)\vee j}^{\lfloor n\tau \rfloor} d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\Delta_2^{1/2}(j,h)\eta_j\right|^q$$
$$= o(|\tau - s|^q). \tag{A.12}$$

By (A.11) and (A.12), Eq. (A.8) holds. Next consider (b). By Doob's inequality,

$$E\left|\sup_{0\le\tau\le 1} \frac{I_2(\lfloor n\tau \rfloor, H)}{k_n}\right|^2$$
$$\le 4n^{-4/\alpha} \sum_{j=-\infty}^{n} E\left|\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee j}^{n} d_{t-j}d_{t-j+h}\right)\varepsilon_{j-h}\Delta_2^{1/2}(j,h)\eta_j\right|^2$$
$$\le C n^{-4/\alpha} \sum_{j=-\infty}^{n}\sum_{h=H+1}^{\infty}\left(\sum_{t=1\vee(j+1)}^{n} (t-j)^{-\beta}(t-j+h)^{-\beta}\right)^2$$
$$\le C n^{-4/\alpha} \sum_{t=1}^{n}\sum_{t'=1}^{n}\sum_{j=0}^{\infty}\sum_{h=H+1}^{\infty} (t+j)^{-\beta}(t'+j)^{-\beta}(t+j+h)^{-\beta}(t'+j+h)^{-\beta}$$
$$+ C n^{-4/\alpha} \sum_{j=1}^{n}\sum_{h=H+1}^{\infty}\sum_{t=j+1}^{n}\sum_{t'=j+1}^{n} (t-j)^{-\beta}(t'-j)^{-\beta}(t-j+h)^{-\beta}(t'-j+h)^{-\beta}$$
$$=: \Sigma_1 + \Sigma_2. \tag{A.13}$$

It is not difficult to see, as $n \to \infty$,

$$\Sigma_1 \le C n^{-4/\alpha} n^{4(1-\beta)} \to 0, \tag{A.14}$$

since $\beta > 1 - 1/\alpha$. On the other hand,

$$\Sigma_2 \le C n^{-4/\alpha} \sum_{j=1}^{n}\sum_{h=H+1}^{n}\sum_{t=j+1}^{n}\sum_{t'=j+1}^{n} (t-j)^{-\beta}(t'-j)^{-\beta}(t-j+h)^{-\beta}(t'-j+h)^{-\beta}$$
$$+ C n^{-4/\alpha} \sum_{j=1}^{n}\sum_{h=n}^{\infty}\sum_{t=j+1}^{n}\sum_{t'=j+1}^{n} (t-j)^{-\beta}(t'-j)^{-\beta}(t-j+h)^{-\beta}(t'-j+h)^{-\beta}$$

$$\begin{aligned}
&\le Cn^{-4/\alpha}\sum_{j=1}^{n}\sum_{h=H+1}^{n}\left(\sum_{t=j+1}^{n}(t-j)^{-\beta}(t-j+h)^{-\beta}\right)^2\\
&\quad + Cn^{-4/\alpha-2\beta+1}\sum_{j=1}^{n}\left(\sum_{t=j+1}^{n}(t-j)^{-\beta}\right)^2\\
&\le Cn^{-4/\alpha}\sum_{j=1}^{n}\sum_{h=H+1}^{n}h^{2(-2\beta+1)} + Cn^{-4/\alpha-4\beta+4}\\
&\le Cn^{-4/\alpha}\Big[n^{4(1-\beta)}I(\beta<3/4)+n\log n I(\beta=3/4)\\
&\quad + nH^{-4\beta+3}I(\beta>3/4)\Big]\to 0,
\end{aligned} \tag{A.15}$$

as $n\to\infty$ and $H\to\infty$. Thus (A.8) holds for $2<\alpha\le 4$ with $\beta>1-1/\alpha$. Finally consider (c). The proof is the same as that in (A.13)–(A.15), with $n^{-4/\alpha}$ replaced by n^{-1}. Thus (A.8) also holds for $\alpha>4$ with $\beta>3/4$. This completes the proof. □

Lemma A.3. *Refer to (103) in the proof of Theorem 3.3. When $\alpha=2$ and as $L\to\infty$,*

$$\sup_{0\le\tau\le1} n^{-1}\Lambda_{3n}(L,\tau)=\sup_{0\le\tau\le1} n^{-1}\sum_{j=1-L}^{\lfloor n\tau\rfloor-L}\left(\sum_{t=L+j}^{\lfloor n\tau\rfloor}d_{t-j}^2\right)(\varepsilon_j^2-c_n)\xrightarrow{P}0. \tag{A.16}$$

Proof. Fix a τ, let $a_j=\sum_{t=L+j}^{\lfloor n\tau\rfloor}d_{t-j}^2$, $S_j=\sum_{l=j}^{\lfloor n\tau\rfloor-L}\left(\varepsilon_j^2-c_n\right)$ and $S_{\lfloor n\tau\rfloor-L+1}=0$.

$$\Lambda_{3n}(L,\tau)=\sum_{j=1-L}^{\lfloor n\tau\rfloor-L}a_j\left(S_j-S_{j+1}\right)=a_{1-L}S_{1-L}+\sum_{j=2-L}^{\lfloor n\tau\rfloor-L}\left(a_j-a_{j-1}\right)S_j. \tag{A.17}$$

By Lemma 4.3(a), we have $\sup_{1-L\le j\le\lfloor n\tau\rfloor-L}\left|S_j\right|/n=O_P(1)$. Thus, by (A.17),

$$\begin{aligned}
\frac{1}{n}\Lambda_{3n}(L,\tau)&=O_P(a_{1-L})+\left(\sup_{1-L\le j\le\lfloor n\tau\rfloor-L}\left|S_j\right|/n\right)\sum_{j=2-L}^{\lfloor n\tau\rfloor-L}\left(a_{j-1}-a_j\right)\\
&=O_P(a_{1-L})=o_P(1),\quad \text{as } L\to\infty.
\end{aligned} \tag{A.18}$$

Similarly, we can show that for any $0\le s<\tau\le1$,

$$\begin{aligned}
\frac{1}{n}\left(\Lambda_{3n}(L,\tau)-\Lambda_{3n}(L,s)\right)&=\frac{1}{n}\sum_{j=1-L}^{\lfloor n\tau\rfloor-L}\left(\sum_{\lfloor ns\rfloor+1}^{\lfloor n\tau\rfloor}d_{t-j}^2\right)\left(\varepsilon_j^2-c_n\right)\\
&\quad+\frac{1}{n}\sum_{j=\lfloor ns\rfloor-L+1}^{\lfloor n\tau\rfloor-L}\left(\sum_{t=L+j}^{\lfloor n\tau\rfloor}d_{t-j}^2\right)\left(\varepsilon_j^2-c_n\right)\\
&=o_P(|\tau-s|).
\end{aligned} \tag{A.19}$$

Combining (A.17) and (A.19) yields (A.16) as desired. □

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