ARBITRAGE PRICING OF SINGLE-NAME CREDIT DERIVATIVES

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Abstract. In existing pricing theories, pricing of single-name credit default swaps (CDSs) and their options makes no reference to the prices of defaultable bonds, the underlying assets of those derivatives. Such a pricing practice does not exclude possible arbitrage across bond and CDS markets. In this paper, we introduce a new theory that treats the two markets as one and thus ensures price consistency. The basic building blocks of our theory are risky zero-coupon bonds backed by the coupons of defaultable bonds. We develop a market model for credit derivatives, and the model bears high analogy to the LIBOR market model. Compared with other existing theories, our theory has two distinguished features. First, the recovery rate is no longer required as an input. Second, credit default swaps can be replicated statically by risky bonds and annuities. According to our theory, the introduction of CDSs eliminates recovery-rate risks in underlying defaultable bonds, leaving "early redemption" as the only residual risk.

Key words: credit default swaps, credit default swaptions, defaultable floating-rate notes, recovery rate, LIBOR market model.

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1. Introduction

The ever-increasing liquidity of credit default swaps (CDS) and options on the credit default swaps (hereafter swaptions) in recent years has motivated the development of a number of pricing theories, among which the ones by Schönbucher (2000), Arvanitis and Gregory (2001), Jamshidian (2002) and Hull and White (2002) are better known to practitioners. These several theories, in dealing with swap rate evaluation and swaption pricing, share two major features. First, the calculation of a swap rate is done through equating the present value (PV) of cash flows of the fee leg and of the protection leg, and, second, the pricing of the swaptions is done by using the Black's formula. These theories seem to work well in swaption market, where the CDS' are treated as the underlying and hedging vehicles.

Nevertheless, when holding these theories to careful scrutiny, it is not very difficult to spot two major problems. First, the exogenously input number for the recovery rate is subjective and often not justified. The model thus has not properly price the recovery-rate risk, which makes the marking-to-market value of a CDS unreliable. Second, the pricing formula for CDS does not imply a static replicating strategy using defaultable bonds and/or annuities. The implication is that a CDS can not be used for static hedging of defaultable bonds neither, and the existing theories may not exclude possible arbitrage across CDS and bond markets. The above findings call for an overhaul for the existing theories, in view of the fact that a substantial volume in the CDS market nowadays is relevant to hedging or managing structured credit portfolios, which contain risky bonds as major components.

The root of both problems, as we see, lies in the very definition of risky zero-coupon bonds, the basic building blocks of any pricing model for credit markets. According to the existing definition, the price of a risky zero-coupon bond equals to the product between the price of its corresponding risk-free zero-coupon bond and its survival probability. Zero-coupon bonds so defined, in reality, are neither traded nor replicable, and their prices carry no information about the recovery rate. This observation has motivated the development of a new theory in this paper. Starting from the root of the problems, we redefine risky zero-coupon bonds for a credit name as single cash flows backed by the coupons of corporate bonds of the same credit name. Intuitively, zero-coupon bonds so defined are tradable, and this results in a series of positive consequences: the recovery rate is no longer an exogenous input but is implied by data, a market model can be developed for credit derivatives, and the hedging of the credit derivatives becomes transparent, paralleled to that of the LIBOR derivatives.

Our theory put the pricing of both CDS and risky bonds under the same framework, and it excludes arbitrage across CDS markets and risky bond market. For efficient model implementation, we make use of implied term structure of hazard rate and implied term structure recovery rate. These term structures are implied by those of CDS rates and risky bond yields. The crucial insight we have obtained in this research is that the market of risky bonds is incomplete due to the presence of recovery rate risk, yet the introduction of CDS' makes such risk tradable and ultimately results in the completeness of the extended market of risky bonds and CDS'.

The market model for credit derivatives developed in this paper bears high analogy to the LIBOR market model. For swaption pricing, we have justified the use of a lognormal model for CDS rates which then renders the Black's formula. In applications, we simply superimpose the market model of credit spreads on top of the LIBOR market model. The inputs for the calibration of the market model of spreads are CDS rate volatilities, in addition to CDS rates, risky bond prices, and the term structure of LIBOR. The calibration can be done in a way similar to that of LIBOR market model.

This paper is organized as follows. In §2 we present the definition of risky zero-coupon bonds and explain the intuition behind the definition. In §3 we define risky forward rates and forward spreads. In §4 we deliver swap-rate formulae for two types of credit swaps. §5 is devoted to par CDS rates of both floating-rate and fixed-rate bonds. In §6 we demonstrate the estimation of implied survival rates and implied recovery rates. In §7 we present the market model for credit derivatives. Finally in §8 we conclude. Some technical details are put in appendix.

2. Pricing of risky bonds: a new perspective

Our model is set in the filtered probability space, $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where the filtration satisfies the usual conditions of right-continuity and completeness, and $\mathbb{Q}$ is the risk-neutral measure. All stochastic processes are adapted to $(\mathcal{F}_t)_{t\geq 0}$. Without loss of generality, we model the default time as the first jump time of a Cox process with an intensity (or hazard rate) process, $\lambda(t)$. We make no specific assumption on the recovery rate other than it is stochastic and time-inhomogeneous. For clarity of presentation, we assume that credit spreads and US Treasury yields are independent, and that there is a single seniority across all bonds under a credit name.

We begin with bond pricing. A defaultable coupon bond pays regular coupons until a default occurs or the maturity is reached. In case of a default, the market convention is that a creditor will receive a final payment that consists of a fraction of principal and accrued interest. The schedule of the final payment varies. Without loss of generality, we assume that 1) the final payment is made at the next coupon date following default and 2) the last coupon accrues until the final payment date[1]. Let c be the coupon rate of a risky bond with tenor[2] $[T_m, T_n]$, τ be the default time and R_τ be the recovery rate at the default time. Then, the cash flow at $T_{j+1}, m \leq j \leq n-1$ can be expressed as

$$\Delta T c \mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau (1 + \Delta T c) \mathbf{1}_{\{T_j < \tau \leq T_{j+1}\}},$$

where $\Delta T = 1/(\text{coupon frequency})$ and $\mathbf{1}_{\tau \in \Omega}$ is the indicator function that equals to 1 if $\tau \in \Omega$, or 0 if otherwise. According to the arbitrage pricing theory (APT) (Harrison and Pliska, 1981), the bond is then priced as the risk-neutral expectation of discounted cash flows:

$$\begin{aligned} B^c(t) &= \sum_{j=m}^{n-1} E_t^Q \left[\frac{B(t)}{B(T_{j+1})} \left\{ \Delta T_j c \mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau (1 + \Delta T_j c) \mathbf{1}_{\{T_j < \tau \leq T_{j+1}\}} \right\} \right] + E_t^Q \left[\frac{B(t)}{B(T_n)} \mathbf{1}_{\{\tau > T_n\}} \right] \\ &= \sum_{j=m}^{n-1} P_{j+1}(t) \Delta T_j c E_t^{Q_{j+1}} \left[\mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau \mathbf{1}_{\{T_j < \tau \leq T_{j+1}\}} \right] \\ &\qquad + \sum_{j=m}^{n-1} P_{j+1}(t) E_t^{Q_{j+1}} \left[R_\tau \mathbf{1}_{\{T_j < \tau \leq T_{j+1}\}} \right] + P_n(t) E_t^{Q_n} \left[\mathbf{1}_{\{\tau > T_n\}} \right], \end{aligned} \tag{1}$$

[1]The protection payment made only at a coupon date is a harmless idealization. There are other default payment schedules as well. For example, the last payment may occur at $\tau + 90$ days. Most payment schedules can be accommodated by as adjustment of the recovery rate.

[2]The coupon dates are $\{T_j\}_{j=m+1}^n$. The bond is said to be "forward starting" if $m > 0$.

where, in addition,

$B(t)$ — the money market account of value $\exp(\int_0^t r_s ds)$, where r_t is the risk-free spot rate;

$P_j(t)$ — the time-t price of the risk-free zero-coupon bond of \$1 notional value maturing at T_j, which equals to $E_t^Q[B(t)/B(T_j)]$;

ΔT_j — $T_{j+1} - T_j$, the length of the coupon interval $(T_j, T_{j+1}]$;

$\mathbb{Q}_j$ — the T_j forward measure;

$E_t^Q[\,\cdot\,]$ — the expectation under $\mathbb{Q}$ conditional on $\mathcal{F}_t$.

The expression for payment upon a default, $R_\tau(1 + \Delta T_j c)$, conforms well with real-world practice: the compensation to the creditor is determined by the outstanding principal and the accrued interest of the defaulted bond, which are treated as in the same asset class, and future coupons are not taken into consideration (Schönbucher, 2004). The second line of (1) results from the changes of measures, followed by a regrouping of the PVs of the coupons and the principal. The payout at time T_{j+1} is priced by using the T_{j+1}-forward measure.

Parallel to the US Treasury market, we introduce here the "C-strip" and the "P-strip" of risky zero-coupon bonds that are backed separately by coupons and principals. It is not hard to see that the C-strip zero-coupon bonds should be defined as

$$
\begin{aligned}
\bar{P}_j(t) &= E_t^Q\left[\frac{B(t)}{B(T_j)}\left\{\mathbf{1}_{\{\tau > T_j\}} + R_\tau \mathbf{1}_{\{T_{j-1} < \tau \le T_j\}}\right\}\right] \\
&= P_j(t) E_t^{Q_j}\left[\mathbf{1}_{\{\tau > T_j\}} + R_\tau \mathbf{1}_{\{T_{j-1} < \tau \le T_j\}}\right] \triangleq P_j(t) D_j(t),
\end{aligned}
\tag{2}
$$

while the definition for the "P-strip" zero-coupon bonds is simply the last line of equation (1). Here in (2), the "$\triangleq$" means that a new variable $D_j(t)$ is defined through the equation. Unlike their Treasury counterparts, the risky zero-coupon bonds of the two strips have apparently different cash-flow structures. Given only the prices of risky coupon bonds, we cannot identify the PVs of zero-coupon bonds in either strip unless additional information is provided.

In principle, zero-coupon bonds of the two strips can be generated through marketing the cash flows of the coupons and the principals separately. In reality, however, risky zero-coupon bonds are not traded and direct price information is not available, with occasional exceptions in Japanese market. Nonetheless, the prices of risky zero-coupon bonds of both strips can be backed out from associated coupon bond prices and additional information like the CDS rates. The C-strip zero-coupon bonds, in particular, will play a major role in our model construction.

3. Forward spreads

To understand the product nature of CDS, we need to clarify "risky forward rates" introduced by Brigo (2005). A risky forward rate should be defined as the fair rate on a defaultable loan for a future period of time, say, $(T_j, T_{j+1}]$, that is collateralized by the coupon flows of defaultable bonds of an entity. If a default of the bond occurs before T_j, the contract ceases to exist. If a default occurs between T_j and T_{j+1}, then a recovery value proportional to the recovery rate of the bonds applies. Assume the notional of the loan to be \$1. According to APT, the risky forward rate, denoted as $\hat{f}_j(t)$, must nullify the PV of the cash flows of the risky loan:

$$\begin{aligned} 0 &= E_t^Q\left[\frac{B(t)}{B(T_j)}\mathbf{1}_{\{\tau>T_j\}}\right] - E_t^Q\left[\frac{B(t)}{B(T_{j+1})}(1+\Delta T_j\hat{f}_j(t))(\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}})\right] \\ &= P_j(t)E_t^{Q_j}[\mathbf{1}_{\{\tau>T_j\}}] - P_{j+1}(t)(1+\Delta T_j\hat{f}_j(t))D_{j+1}(t) \\ &\triangleq P_j(t)Q(\tau>T_j) - P_{j+1}(t)(1+\Delta T_j\hat{f}_j(t))D_{j+1}(t), \end{aligned} \tag{3}$$

where $Q(\tau > T_j)$ is the $\mathbb{Q}$ probability of survival until T_j, and it is equal to the $\mathbb{Q}_j$ probability of survival until T_j due to the independence between US Treasury yields and the default probability. Equation (3) gives rise to

$$\begin{aligned} \hat{f}_j(t) =& \frac{1}{\Delta T_j}\left[\frac{P_j(t)}{P_{j+1}(t)}\frac{Q_j(\tau>T_j)}{D_{j+1}(t)} - 1\right] \\ =& \frac{1}{\Delta T_j}\left[\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)}\frac{Q_j(\tau>T_j)}{D_j(t)} - 1\right]. \end{aligned} \tag{4}$$

The two lines in the equation above lead to two alternative expressions of $\hat{f}_j(t)$. The first expression is

$$\hat{f}_j(t) = f_j(t) + \frac{1+\Delta T_j f_j(t)}{\Delta T_j}\left(\frac{E_t^{Q_{j+1}}[(1-R_\tau)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right), \tag{5}$$

where $f_j(t)$ is the default-free forward rate for the period $(T_j, T_{j+1}]$ seen at time t, defined by

$$f_j(t) = \frac{1}{\Delta T_j}\left(\frac{P_j(t)}{P_{j+1}(t)} - 1\right), \quad t \le T_j.$$

The second expression is

$$\hat{f}_j(t) = \bar{f}_j(t) - \frac{1+\Delta T_j\bar{f}_j(t)}{\Delta T_j}\left(\frac{E_t^{Q_j}\left[R_\tau\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}\right]}{D_j(t)}\right), \tag{6}$$

where $\bar{f}_j(t)$ is the "defaultable effective forward rate" (Schönbucher, 2000), defined by

$$\bar{f}_j(t) = \frac{1}{\Delta T_j}\left(\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)} - 1\right), \quad t \le T_j.$$

Note that $\bar{f}_j(t)$ should be understood as the effective rate of return over $(T_j, T_{j+1}]$ provided that no default occurs until T_{j+1}. It is pointed out in Brigo (2004) that $\bar{f}_j(t)$ in general does not link directly to a financial contract. Putting (5) and (6) together, we have the order

$$f_j(t) \le \hat{f}_j(t) \le \bar{f}_j(t). \tag{7}$$

Note that $\hat{f}_j(t)$ achieves the upper bound and the lower bound when $R_\tau = 0$ and $R_\tau = 1$, respectively. The bounds on $\hat{f}_j(t)$ should be regarded as no-arbitrage constraints.

Here we make an enhanced statement to the "non-separability" by Duffie and Singleton (1999), which means that the hazard rate and the loss rate cannot be determined from bond prices alone. In view of (4), we can say that complete term structures of the survival probability and the recovery rate, $Q(\tau > T_j)$ and $E_t^Q[R_\tau | T_{j-1} < \tau \le T_j], j = 1, 2, \ldots,$ can be uniquely determined from the term structures of $\hat{f}_j(t)$ and $\bar{f}_j(t)$, which in fact are informatively equivalent to the term structures of risky-bond yields and CDS rates.

Intuitively, a "forward spread" is defined as the difference between a risky forward rate and its corresponding risk-free forward rate:

$$S_j(t) = \hat{f}_j(t) - f_j(t), \quad j = 1, 2, \ldots .$$

From equation (5), we obtain

$$S_j(t) = (1 + \Delta T_j f_j(t)) \frac{E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}]}{\Delta T_j D_{j+1}(t)} \triangleq (1 + \Delta T_j f_j(t)) H_j(t). \tag{8}$$

According to its definition, $H_j(t)$ can be interpreted as the "expected loss per risky dollar over $(T_j, T_{j+1}]$". Equation (8) can be rewritten as

$$1 + \Delta T_j \hat{f}_j(t) = (1 + \Delta T_j f_j(t))(1 + \Delta T_j H_j(t)), \quad j = 1, 2, \ldots .$$

By comparing definitions, we can say that $H_j(t)$ is the discrete-tenor version of the "mean loss rate" introduced in Duffie and Singleton (1999).

4. Two kinds of default protection swaps

A default protection swap consists of a fee leg (or premium leg) and a protection leg. Before the default of the reference entity, the protection buyer pays the protection seller a string of fees at regular time intervals. Upon default, the protection buyer either delivers the bond to the protection seller in exchange for par (so-called physical delivery), or receives from the protection seller a payment that is equal to the loss incurred (cash settlement). In this section, we consider two kinds of swaps: swaps for fixed-rate bonds and swaps for floating-rate bonds. Without loss of generality, we assume the notional value of the bonds to be \$1, and the coupon rate to be c and LIBOR, $f_j(T_j)$, respectively. For the swap on the fixed-rate bond, the protection payment is $(1-R_\tau)(1+\Delta T c)$, while for the swap on the floating-rate bond, the protection payment is $(1-R_\tau)(1+\Delta T_j f_j(T_j))$. The payments simply reflect the loss to the bond holders in case of a default[3]. Note that the swaps of the second kind depend only on the default status of the reference entity.

We proceed to the computation of fair default swap rates. Let us denote a swap rate by $\bar{s}$. In the CDS markets, the contractual cash flows of the fee leg (for the protection of \$1 notional) are typically

$$\bar{s}\Delta T_j[\mathbf{1}_{\{\tau>T_{j+1}\}} + \frac{(\tau - T_j)}{\Delta T_j}\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}], \quad j = 1, 2, \dots ,$$

i.e., in case of a default occurred between T_j and T_{j+1}, the protection buyer makes the final payment that is proportional to the time elapsed between the last fee payment and the default. From the financial engineering point of view, such a fee specification is troubling because the cash flow cannot be synthesized at the initiation of the swap contract. To make the swap contract replicable, we propose a slight modification to the fee leg: we let the cash flows be generated from the payouts of risky zero-coupon bonds:

$$\bar{s}\Delta T_j[\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}].$$

This new definition only changes the last piece of fee payment after default. The value of the fee leg is now

$$\begin{aligned} PV_{fee} &= \bar{s}\sum_{j=m}^{n-1}\Delta T_j P_{j+1}(t)E_t^{Q_{j+1}}[\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}] \\ &= \bar{s}\sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t). \end{aligned} \tag{9}$$

The fee leg is now a tradable annuity, analogous to the fixed leg of default-free swaps.

For a swap of the first kind, the cash flow of the protection seller can be written as

$$V_{prot} = (1+\Delta T c)\sum_{j=m}^{n-1}(1-R_\tau)\mathbf{1}_{\{T_j\le\tau\le T_{j+1}\}}.$$

[3]The quantity $(1+\Delta T_j f_j(T_j))$ can be regarded as “a constant dollar” seen at time T_j.

The PV of the protection payment is then

$$PV_{prot} = (1 + \Delta T c) \sum_{j=m}^{n-1} P_{j+1}(t) E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j \le \tau \le T_{j+1}\}}]. \tag{10}$$

By equating the fee leg to the protection leg and making use of (8), we obtain

$$\begin{aligned} \bar{s}_1 &= (1 + \Delta T c) \frac{\sum_{j=m}^{n-1} P_{j+1}(t) E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j \le \tau \le T_{j+1}\}}]}{\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)} \\ &= (1 + \Delta T c) \sum_{j=m}^{n-1} \bar{\alpha}_j H_j(t), \end{aligned} \tag{11}$$

where

$$\bar{\alpha}_j(t) = \frac{\Delta T_j \bar{P}_{j+1}(t)}{\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)}.$$

Note that the case $c = 0$ corresponds to a prototypical default swap, which only depends on the default status of the reference entity and is traded with liquidity.

For swaps of the second kind, the fair swap rate can be derived analogously as

$$\bar{s}_2 = \frac{\sum_{j=m}^{n-1} P_{j+1}(t)(1 + \Delta T_j f_j(t)) E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j \le \tau \le T_{j+1}\}}]}{\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)} = \sum_{j=m}^{n-1} \bar{\alpha}_j S_j(t). \tag{12}$$

Here, we have made use of the independence between credit spreads and the US Treasury yields, and the martingale property of the forward rate: $E_t^{Q_{j+1}}[f_j(T_j)] = f_j(t)$. Note that a one-period CDS rate reduces to a risky forward rate.

Compared with the existing CDS rate formula (e.g., Schönbucher, 2004; Brigo, 2005), the above two formulae make no reference to the recovery rate, and they are analogous to the swap rate formula in LIBOR markets. Note that in reality, CDS rates, instead of forward spreads, are directly observable. Through the formulae, we gain valuable insights into the CDS rates.

5. Par CDS rates

A par CDS rate is a spread that, when added to a corresponding par rate of default-free bond, yields the coupon rate of a risky par bond. We will derive the par CDS rates for both floating rate bonds and fixed-rate bonds.

Typically, a defaultable floating-rate bond (which is also called a defaultable floater) pays LIBOR plus a credit spread, denoted by X, until a default occurs, when a holder may obtain some recovered value of both principal and coupon. Hence, the cash flow of a defaultable floater at T_{j+1} can be written as

$$CF_{j+1} = (1 + \Delta T_j(f_j(T_j) + X))\left(\mathbf{1}_{\{\tau > T_{j+1}\}} + R\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}\right) - \mathbf{1}_{\{\tau > T_{j+1}, j+1 < n\}}.$$

The question here is: if the floater is to be priced at par, what should be the fair spread rate X? To answer this question, we imagine that the holder of the floater is also "long" a protection swap of the second kind. Then, his/her cash flow at time T_{j+1} is

$$\begin{aligned} CF_{j+1} = \Delta T_j f_j(T_j)\mathbf{1}_{\{\tau > T_j\}} + \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}} + \mathbf{1}_{\{\tau > T_n, j+1 = n\}} \\ + (X - \bar{s}_2)(\mathbf{1}_{\{\tau > T_j\}} + R\mathbf{1}_{\{T_{j-1} < \tau \le T_j\}}). \end{aligned} \tag{13}$$

The first line in (13) gives the cash flow of a rolling-forward CD that lasts until $T_{j_\tau} \wedge T_n$, where T_{j_τ} is the first fixing date after default, and such cash flows represent those of a par bond (that matures at $T_{j_\tau} \wedge T_n$). It then becomes clear that, for the defaultable floater to be priced at par, there must be

$$X = \bar{s}_2,$$

i.e., the default swap rate equals nothing else but the credit spread!

Given the clear relationship between a defaultable floater and a CDS, we come up with the following hedging strategy for swaps of the second kind: once such a swap is written, the hedger goes "long" a default-free floater and goes "short" a defaultable floater, both at par. Then, the net cash flow at any fixing date is zero.

Next, we derive the par CDS rate for a corresponding fixed-rate bond with tenor $(T_m, T_n]$. Again we let X denote the par CDS rate for the fixed-rate bond. Then, X can be determined by equating the PVs of fixed-rate and floating-rate par bonds:

$$\begin{aligned} (R_{m,n}(t) + X)\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t) &= \sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)[f_j(t) + \bar{s}_2] \\ &= \sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)\hat{f}_j(t), \end{aligned}$$

where the PVs of the principals have been canceled, and $R_{m,n}(t)$ represents the corresponding swap rate (i.e., par rate) in LIBOR markets, defined by

$$R_{m,n}(t) = \sum_{j=m}^{n-1} \alpha_j f_j(t), \qquad \alpha_j = \frac{\Delta T_j P_{j+1}(t)}{\sum_{k=m}^{n-1} \Delta T_k P_{k+1}(t)}.$$

It follows that

$$X = \sum_{j=m}^{n-1} \bar{\alpha}_j \hat{f}_j(t) - R_{m,n}(t)$$
$$= \bar{s}_2 + \sum_{j=m}^{n-1} (\bar{\alpha}_j - \alpha_j) f_j(t).$$

Hence, the so-called "credit spread" is different for floating-rate bonds and fixed-rate bonds, and a par CDS rate for fixed-rate bonds is close to $\bar{s}_2$ but not necessarily $\bar{s}_1$ for $c = 0$.

It can be verified that the short position of the default swap on the risky coupon bond with coupon rate c can be hedged by

(1) being "short" the risky coupon bond,

(2) being "long" a risk-free floater at par, and

(3) being "long" $(c - \bar{R}_{m,n}(t))$ units of the risky annuity, $\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)$.

Here, $\bar{R}_{m,n}(t) = R_{m,n}(t) + X$ is the risky par yield.

6. Implied survival curve and recovery-rate curve

In reality, neither $\{\hat{f}_j(t)\}$ or $\{\bar{f}_j(t)\}$ is observable. For CDS pricing and other applications, it is more convenient to make use of the term structures of forward hazard rate and forward recovery rate. We define the forward hazard rate for $(T_{j-1}, T_j]$ seen at time t by

$$\lambda_j(t) = \frac{1}{\Delta T_{j-1}} \left(\frac{Q(\tau > T_{j-1})}{Q(\tau > T_j)} - 1 \right), \tag{14}$$

and the forward recovery rate for the same period by

$$R_j(t) = E_t^Q \left[R_\tau | T_{j-1} < \tau \leq T_j \right], \quad j = 1, 2, \ldots . \tag{15}$$

The survival probability, $Q(\tau > T_j)$, relates to the hazard rates by

$$Q(\tau > T_j) = \prod_{k=1}^{j-1} (1 + \Delta T_k \lambda_k)^{-1}. \tag{16}$$

The standard market practice is to back out the survival probabilities from CDS of various maturities, assuming a constant recovery rate (of 40%). Instead of doing the same thing with our model, we consider backing out simultaneously the implied hazard rates and recovery rates from CDSs and, in addition, corporate bond prices. We choose Citigroup as the credit name for a demonstration, as there is relatively richer credit information on this company. A snapshot of market quotations is provided in Tables 1 and 2, where the currency is US dollars (USD), the CDS rates are for $c = 0$, and the bond prices are "clean". To build the risk-free discount curve in USD, we have used LIBOR rates and futures implied rates up to two years, and swap rates from 2 to 20 years. The interest-rate information is provided in Table 3. Figure 1 presents the forward-rate curve constructed using the yield data in Table 3.

Table 1. Citigroup CDS rates (28/7/2005, Bloomberg)

Maturity	1Y	3Y	5Y	10Y
Rates	0.07%	0.13%	0.19%	0.33%

Table 2. Prices of benchmark Citigroup bonds (28/7/2005, Bloomberg)

Maturity	Frequency	Coupon	Price
22/2/2010	Semi-annual	4.125%	98.123
1/10/2010	Semi-annual	7.25%	114.563
7/5/2015	Semi-annual	4.875%	97.563
18/5/2010	Quarterly	US LIB+15bps	99
16/3/2012	Quarterly	US LIB+12.5bps	99.80
5/11/2014	Quarterly	US LIB+28bps	100.501

Table 3. USD yield data (28/7/2005, Bloomberg)

LIBOR	3 mth	3.6931%
	6 mth	3.8435%
	12 mth	4.1731%
Swap	2Y	4.3200%
	3Y	4.3840%
	4Y	4.4330%
	5Y	4.4690%
	7Y	4.5365%
	10Y	4.6290%
	12Y	4.6905%
	15Y	4.7630%
	20Y	4.8320%

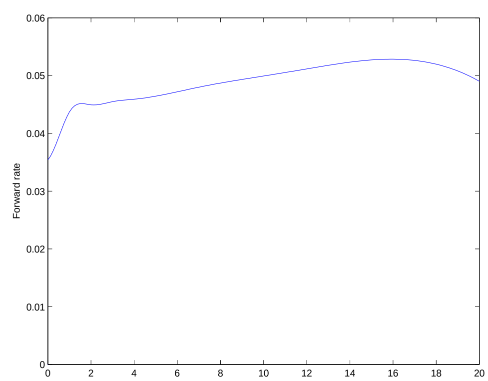

FIGURE 1. USD Forward rates (28/07/2005).

We determine $\{\lambda_j, R_j\}$ through reproducing the CDS rates and the bond prices of Citigroup by the swap-rate formula (11) and the bond formula (1), respectively. Because the problem is under determined, we have adopted cubic-spline and linear interpolation for the hazard rates and the recovery rates, respectively, and imposed additional smoothness regularization. In our search algorithm, we taken various initial guesses with $\lambda_j = 0$ and $0.0 \leq R_j \leq 0.6, \forall j$. Most of the times, the search ends up in one of the two solutions, depicted in Figure 2 - 4, depending on the closeness of the initial recovery rate to either $R_0 = 0.0$

or $R_0 = 0.4$. Existence of more than one solution reflects the ill-posed nature of calibration problem, particularly with regard to determining the recovery rate. Our experiences suggest the market quotations of CDS rates are pretty much the hazard rates corresponding to the zero recovery rate.

We remark here the ill-posed nature of calibration problem is largely intrinsic. This is due to the insensitivity of the bond and CDS prices with respect to the change of the recovery rates[4], particularly for a good credit name. As a consequence of the ill-posedness, implied recovery rates and default rates may depend on initial guesses, as are shown in Figure 2 and 3. To nail down to a single solution, additional regularizations (based on financial or mathematical considerations) are needed. On the other hand, the ill-posedness may not be a big concern. As is shown in Figure 4, risky forward rates and risky discount curve demonstrate noticeable stability. This is due to the complimentary effect between the hazard rates and the recovery rates in calibration.

7. Credit default swaptions and an extended market model

For either the protection buyer or seller, an open position can be closed by either going "short" or going "long" of the same swap. Because the protection legs are exactly offset, the profit or loss for the pair of transactions comes from the difference of the fee legs, which is

$$(\bar{s}_{m,n}(t) - \bar{s}_{m,n}(0)) \sum_{j=m}^{n-1} \Delta T_j \bar{P}_j(t), \tag{17}$$

where $\bar{s}_{m,n}(t)$ is the prevailing CDS rate, either $\bar{s}_1$ or $\bar{s}_2$, seen at time t.

A credit swaption, meanwhile, is a contract that gives its holder the right, not the obligation, to enter into a forward-starting swap at time $T \leq T_m$ with a predetermined swap rate, $\bar{s}^*$. Apparently, the profit/loss at maturity T of the swaption is

$$(\bar{s}_{m,n}(T) - \bar{s}^*)^+ \sum_{j=m}^{n-1} \Delta T_j \bar{P}_j(T). \tag{18}$$

The standard practice in markets is to price the swaption by using Black's formula. Effectively, the markets have made use of a *default swap measure*[5], $\mathbb{Q}^S$, defined by the Radon-Nykodim derivative

$$\left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B^S(t)}{B^S(0)} \bigg/ \frac{B(t)}{B(0)}\ , \tag{19}$$

[4] The insensitivity is also demonstrated in Brigo (2005).

[5] Schönbucher (2004) calls it a survival measure for the case of $R = 0$.

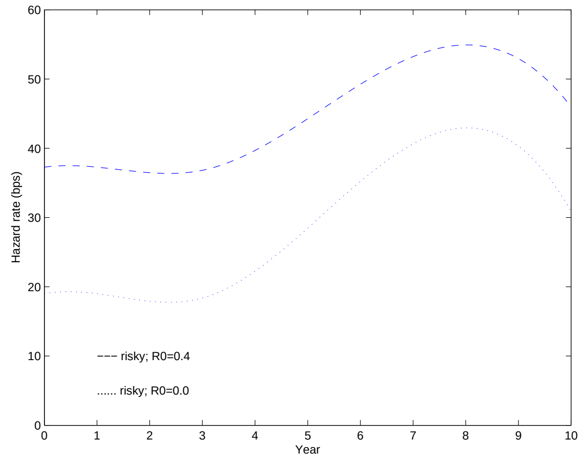

FIGURE 2. Implied hazard rates.

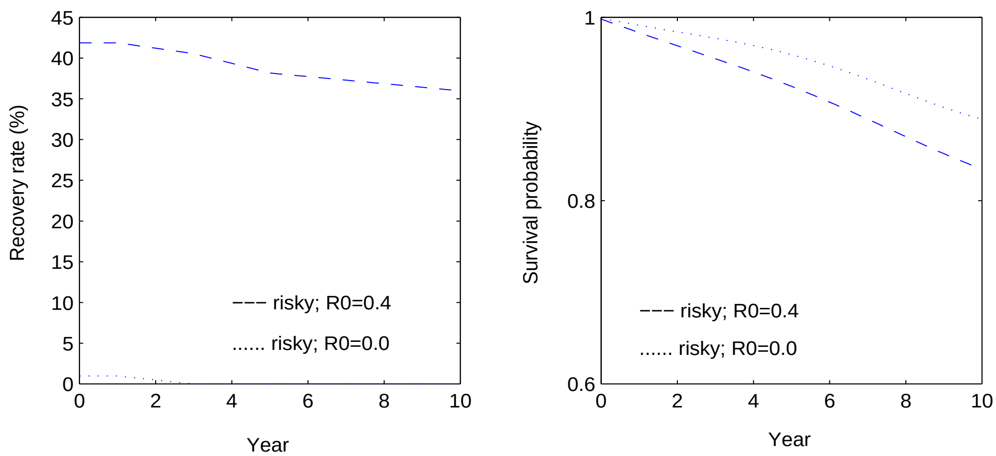

FIGURE 3. Implied default rates recovery rates.

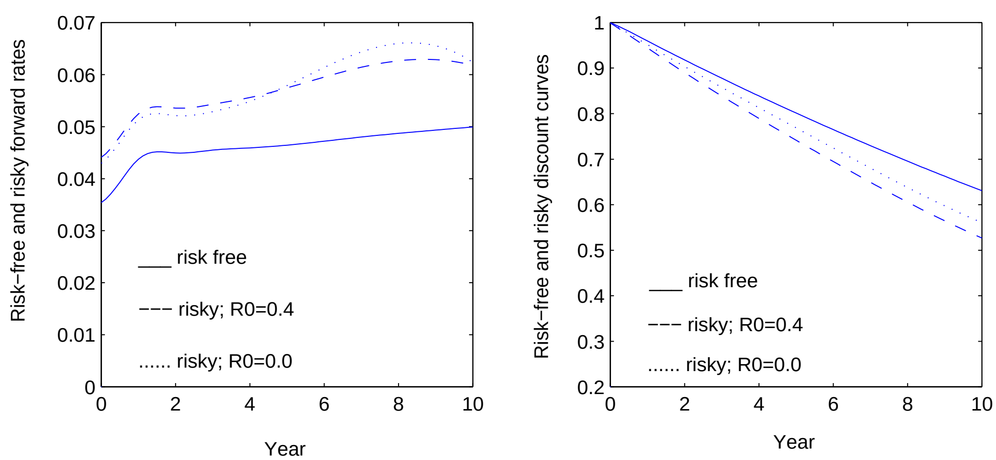

FIGURE 4. Forward rates and discount curves.

where

$$B^S(t) = \sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t).$$

Note that $\mathbb{Q}^S$ is absolutely continuous w.r.t. $\mathbb{Q}$, but not equivalent to it in the case of $B^S(t) = 0$. According to expression (11) or (12), a default swap rate is an asset price relative to the numeraire, $B^S(t)$, and hence is a martingale under $\mathbb{Q}^S$. In the credit derivative markets, $\bar{s}_{m,n}(t)$ is assumed to follow a lognormal process:

$$d\bar{s}_{m,n}(t) = \bar{s}_{m,n}(t)\bar{\gamma}_{m,n}\, dW_t^S, \tag{20}$$

where W_t^S, to be defined shortly, is a one-dimensional Brownian motion under $\mathbb{Q}^S$ and $\bar{\gamma}_{m,n}$ is the swap-rate volatility. The lognormality assumption, (22), leads readily to Black's formula for credit swaptions:

$$C = B^S(t)[\bar{s}_{m,n}(t)N(d_1) - \bar{s}^* N(d_2)], \tag{21}$$

with

$$d_{1,2} = \frac{\ln(\bar{s}_{m,n}(t)/\bar{s}^*) \pm \frac{1}{2}\bar{\gamma}_{m,n}^2(T-t)}{\bar{\gamma}_{m,n}\sqrt{T-t}}.$$

A hedging strategy follows from Black's formula (21): at time $t < T$, the hedger

maintains $N(d_1)$ units of the credit default swap; and

maintains $\bar{s}_{m,n}(t)N(d_1) - \bar{s}^* N(d_2)$ units of the annuity.

Next we show that the above swaption pricing approach can be embedded into the framework of market models. On top of the standard market model with the risk-free forward rates (Brace, Gatarek and Musiela, 1997; Jamshidian, 1997; and Miltersen, Sandmann and Sondermann, 1997):

$$\begin{aligned} \frac{df_j(t)}{f_j(t)} &= \mu_j(t)dt + \gamma_j(t) \cdot d\mathbf{W}_t, \\ \mu_j(t) &= \gamma_j(t) \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)}{1 + \Delta T_k f_k(t)} \gamma_k(t), \end{aligned} \tag{22}$$

where $\mathbf{W}_t$ is a multi-dimensional Brownian motion under $\mathbb{Q}$, we impose a lognormal model, depending on applications, with either spreads:

$$\frac{dS_j(t)}{S_j(t)} = \mu_j^S(t)dt + \gamma_j^S(t) \cdot d\mathbf{W}_t, \tag{23}$$

or mean loss rates:

$$\frac{dH_j(t)}{H_j(t)} = \mu_j^H(t)dt + \gamma_j^H(t) \cdot d\mathbf{W}_t. \tag{24}$$

In (24), the drift term is a function of $\gamma_k^H, k \le j$:

$$\mu_j^H(t) = \gamma_j^H(t) \sum_{k=1}^{j} \frac{\Delta T_k H_k(t)}{1 + \Delta T_k H_k(t)} \gamma_k^H(t). \tag{25}$$

The drift term, μ_j^S, takes a more complex form:

$$\begin{aligned} \mu_j^S(t) &= \frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)} \mu_j + \left(\frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)} \gamma_j - \sigma_{j+1}^D(t) \right) \left(\gamma_j^S(t) - \frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)} \gamma_j(t) \right), \\ \sigma_{j+1}^D(t) &= -\sum_{k=1}^{j} \frac{\Delta T_k S_k(t)}{1 + \Delta T_k \hat{f}_k(t)} \left(\gamma_j^S(t) - \frac{\Delta T_j}{1 + \Delta T_j f_j(t)} \gamma_j(t) \right). \end{aligned} \tag{26}$$

The derivations are left in the appendix.

Next, we proceed to the pricing of swaptions under the extended market model. We take the second kind of swaps as an example. The swap rate is given by (12). By Ito's lemma, we can derive an approximate swap-rate process as follows:

$$\begin{aligned} d\bar{s}_{m,n}(t) &= \sum_{j=m}^{n-1} \frac{\partial \bar{s}_{m,n}(t)}{\partial S_j} S_j(t) \gamma_j^S \cdot d\mathbf{W}_t^S \\ &= \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \frac{\partial \bar{s}_{m,n}(t)}{\partial S_j} \frac{S_j(t)}{\bar{s}_{m,n}(t)} \gamma_j^S \cdot d\mathbf{W}_t^S \\ &\approx \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \frac{\partial \bar{s}_{m,n}(0)}{\partial S_j} \frac{S_j(0)}{\bar{s}_{m,n}(0)} \gamma_j^S \cdot d\mathbf{W}_t^S \\ &= \bar{s}_{m,n}(t) \bar{\gamma}_{m,n} \cdot d\mathbf{W}_t^S, \end{aligned} \tag{27}$$

where

$$\bar{\gamma}_{m,n} \triangleq \sum_{j=m}^{n-1} \bar{\omega}_j \gamma_j^S, \quad \bar{\omega}_j \triangleq \frac{\partial \bar{s}_{m,n}(0)}{\partial S_j} \frac{S_j(0)}{\bar{s}_{m,n}(0)} \approx \bar{\alpha}_j \frac{S_j(0)}{\bar{s}_{m,n}(0)}, \tag{28}$$

and $\mathbf{W}_t^S$ is a multi-dimensional Brownian motion under $\mathbb{Q}^S$, defined by

$$d\mathbf{W}_t^S = d\mathbf{W}_t - \sum_{j=m}^{n-1} \bar{\alpha}_j \bar{\sigma}_{j+1}(t) dt,$$

and $\bar{\sigma}_{j+1}$ is the volatility of $\bar{P}_{j+1}(t)$. The lognormal process for swap rates, (27), justifies the use of Black's formula, (21), for swaptions. Note that, in the context of LIBOR market model, the approximations made in (27) and (28) are known to be accurate enough for applications and have been justified with rigor (Brigo *et al.*, 2004). The expressions in (27) describe the relation between forward spread volatilities and swap-rate volatilities, which can be used in practice to gauge the relative price richness/cheapness of a swaption. The model for spreads, either (23) or (24), can be calibrated to the implied volatilities of the default swaptions using the quadratic programming technology developed by Wu (2003) for market model calibrations.

Models more comprehensive than (23) or (24) can be developed by including other risk dynamics, like jumps, stochastic volatilities and even correlations among multiple credit names. Such developments will be largely parallel to existing extensions to the standard market model. Brigo (2005) and especially Schönbucher (2004) have made several extensions using swap rates as state variables.

8. Conclusions

In this paper, we rewrite the theory of Schönbucher (2000) for single-name credit derivatives so that credit default swaps become replicable with tradable bonds and annuities, and the recovery rate drops out from the list of inputs. The definitions of risky forward rates and forward spreads are clarified. We develop a market model with the forward spreads, and justify the use of the Black's formula for swaptions. In applications, the market model for the spreads is directly superimposed on the LIBOR market model, and the former can be conveniently calibrated to CDS rate curve, risky bond prices and implied volatilities of swaptions. The market model for spreads also has the capacity to accommodate correlations among multiple credit names by using copulas. This is left for future research.

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Appendix A. Details of some derivations

Derivation of $\hat{f}_j(t)$:

The first line of (4) leads to

$$\begin{aligned}\hat{f}_j(t) =& \frac{1}{\Delta T}\left[\frac{P_j(t)}{P_{j+1}(t)} - 1\right] + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left(\frac{Q_j(\tau > T_j)}{D_{j+1}(t)} - 1\right)\\ =& f_j(t) + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left(\frac{E_t^{Q_j}[\mathbf{1}_{\{\tau>T_j\}}] - E_t^{Q_{j+1}}[\mathbf{1}_{\{\tau>T_{j+1}\}} + R\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right)\\ =& f_j(t) + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left(\frac{E_t^{Q_{j+1}}[(1-R)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right)\\ =& f_j(t) + \frac{1+\Delta T_j f_j(t)}{\Delta T}\left(\frac{E_t^{Q_{j+1}}[(1-R)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right).\end{aligned} \tag{A.1}$$

In the third row of the equation, we have again used the independence between the hazard rate and Treasury yields. The second line of equation (4) leads to

$$\begin{aligned}\hat{f}_j(t) =& \frac{1}{\Delta T}\left[\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)} - 1\right] - \frac{1}{\Delta T}\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)}\left[1 - \frac{Q_j(\tau > T_j)}{D_j(t)}\right]\\ =& \bar{f}_j(t) - \frac{1+\Delta T_j \bar{f}_j(t)}{\Delta T_j}\left(\frac{E_t^{Q_j}\left[R\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}\right]}{D_j(t)}\right). \qquad \square\end{aligned} \tag{A.2}$$

Derivation of the drift term of $H_j(t)$:

According to its definition, we have

$$\begin{aligned}H_j(t) =& \frac{1}{\Delta T_j}\left(\frac{1+\Delta T_j \hat{f}_j(t)}{1+\Delta T_j f_j(t)} - 1\right)\\ =& \frac{1}{\Delta T_j}\left(\frac{\frac{\bar{P}_j}{\bar{P}_{j+1}}\cdot\frac{Q_j(\tau>T_j)}{D_j}}{\frac{P_j}{P_{j+1}}} - 1\right)\\ =& \frac{1}{\Delta T_j}\left(\frac{D_j}{D_{j+1}}\cdot\frac{Q_j(\tau > T_j)}{D_j} - 1\right).\end{aligned} \tag{A.3}$$

For simplicity, we denote $\beta_j = \frac{Q_j(\tau>T_j)}{D_j}$ and make a very weak assumption that β_j is time independent. In fact, if the hazard rate, λ, and recovery rate, R, are assumed to be time independent, we can derive

$$\frac{Q_j(\tau > T_j)}{D_j} = \frac{1}{1 + \bar{R}(e^{\lambda \Delta T_{j-1}} - 1)}, \tag{A.4}$$

which is indeed time-independent. Here $\bar{R} = E^{Q_j}[R]$. It then follows that

$$\begin{aligned} dH_j(t) &= \frac{\beta_j}{\Delta T_j} d\left(\frac{D_j}{D_{j+1}}\right) \\ &= \frac{\beta_j}{\Delta T_j}\frac{D_j}{D_{j+1}}[\sigma_j^D - \sigma_{j+1}^D] \cdot [d\mathbf{W}_t - \sigma_{j+1}^D dt] \\ &= \frac{1 + \Delta T_j H_j}{\Delta T_j}[\sigma_j^D - \sigma_{j+1}^D] \cdot [d\mathbf{W}_t - \sigma_{j+1}^D dt], \end{aligned} \tag{A.5}$$

where σ_j^D is the percentage volatility of $D_j(t)$. Define the volatility of H_j by

$$\gamma_j^H \triangleq \frac{1 + \Delta T_j H_j}{\Delta T_j}[\sigma_j^D - \sigma_{j+1}^D].$$

By recursion, we have

$$\begin{aligned} \sigma_{j+1}^D &= \sigma_j^D - \frac{\Delta T_j H_j}{1 + \Delta T_j H_j}\gamma_j^H \\ &= \sigma_1^D - \sum_{k=1}^{j} \frac{\Delta T_k H_k}{1 + \Delta T_k H_k}\gamma_k^H. \end{aligned} \tag{A.6}$$

In detail,

$$\sigma_1^D = \bar{\sigma}_1 - \sigma_1,$$

where $\bar{\sigma}_1$ and σ_1 are the volatilities of $\bar{P}_1$ and P_1, respectively. Unless a default is imminent, we may comfortably put $\sigma_1^D = 0$ and hence obtain an expression of σ_{j+1}^D in terms of $\gamma_k^H, k \le j$. The drift term of H_j is simply

$$\mu_j^H = -\gamma_j^H \sigma_{j+1}^D. \qquad \square \tag{A.7}$$

Derivation of the drift term of $S_j(t)$:

The drift and volatility terms of S_j can now be derived with ease. Since

$$S_j = (1 + \Delta T_j f_j)H_j,$$

we have

$$\begin{aligned} dS_j &= \Delta T_j H_j df_j + (1 + \Delta T_j f_j) dH_j + \Delta T_j df_j \ dH_j \\ &= \Delta T_j H_j f_j(\mu_j dt + \gamma_j \cdot d\mathbf{W}_t) + (1 + \Delta T_j f_j)H_j(\mu_j^H dt + \gamma_j^H \cdot d\mathbf{W}_t) + \Delta T_j f_j H_j \gamma_j \gamma_j^H dt \\ &= S_j\left[\left(\frac{\Delta T_j f_j}{1 + \Delta T_j f_j}(\mu_j + \gamma_j \gamma_j^H) + \mu_j^H\right) dt + \left(\frac{\Delta T_j f_j}{1 + \Delta T_j f_j}\gamma_j + \gamma_j^H\right) \cdot d\mathbf{W}_t\right] \\ &\triangleq S_j(\mu_j^S dt + \mu_j^S \cdot d\mathbf{W}_t), \end{aligned} \tag{A.8}$$

where μ_j^S and γ_j^S relate to μ_j^H and γ_j^H through

$$\begin{aligned} \mu_j^S &= \mu_j^H + \frac{\Delta T_j f_j}{1 + \Delta T_j f_j}(\mu_j + \gamma_j \gamma_j^H), \\ \gamma_j^S &= \gamma_j^H + \frac{\Delta T_j f_j}{1 + \Delta T_j f_j}\gamma_j. \end{aligned} \tag{A.9}$$

By substituting the above expressions for μ_j^H and γ_j^H in (A.6) and (A.7), we obtain (26). $\square$