MATH 1003 Review: Part 4. Integration

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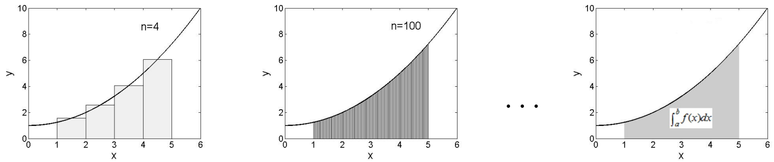
Big Picture

- $F(x)$ is an anti-derivative of $f(x)$, if $F'(x) = f(x)$.
- Indefinite integral (L.25,Ch.13-1)

$$\int \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{differential}} = \underbrace{F(x)}_{\text{anti-derivative}} + \underbrace{C}_{\text{constant}}.$$

- Definite integral: $\int_a^b f(x)dx$ - signed area (L.27,Ch.13-4)



- Fundamental theorem of calculus (L.28,Ch.13-5):

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

Indefinite Integrals

Basic formulas (easy) (L.25,Ch.13-1):

- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, where n is a real number ($n \neq -1$).
- $\int e^x dx = e^x + C$
- $\int \dfrac{1}{x} dx = \ln|x| + C$, where $x \neq 0$.

Basic operations (easy) (L.25,Ch.13-1):

- $\int kf(x)dx = k\int f(x)dx$, where k is a constant.
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Indefinite Integrals

Integration by Substitution (medium) (L.27,Ch.13-2):

- Formula:

$$\int f(x)dx \overset{u=u(x)}{=} \int g(u) \underbrace{u'(x)dx}_{du} = \int \underbrace{g(u)}_{\text{anti-derivative obtainable}} du$$

- General form in applications of the method:

$$\int f(x^p)x^{p-1}dx, \quad \int f(\ln x)\frac{dx}{x}, \quad \int x^{+\text{integer}} \cdot (ax+b)^q dx, \cdots$$

- Examples:

$$\int x^3\sqrt{x^4+1}dx, \quad \int xe^{-x^2}dx, \quad \int \frac{dx}{x\ln x}, \quad \int x^2(3x+2)^{-1/3}dx$$

Indefinite Integrals

Integration by Parts (difficult) (L.31,Ch.14-3):

- Formula:

$$\int u\underbrace{v'dx}_{dv} = uv - \int v\underbrace{u'dx}_{du}$$

- General form in applications of the method:
  - To calculate $\int x^n e^{ax}dx$, we need

$$u = x^n, \quad v'dx = e^{ax}dx \Rightarrow v = \frac{1}{a}e^{ax};$$

  - To calculate $\int x^p(\ln x)^q dx$, we need

$$u = (\ln x)^q, \quad v'dx = x^p dx \Rightarrow v = \frac{1}{p+1}x^{p+1}.$$

- Examples (the method may be applied repeatedly):

$$\int xe^{-2x}dx, \quad \int x^2e^{-2x}dx, \quad \int (\ln x)^2x^5dx$$

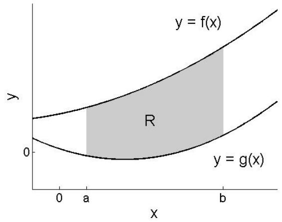
Definite Integrals

- Area between two curves(L.29,Ch.14-1):

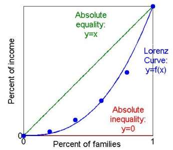
$$R = \int_a^b (f(x) - g(x))dx$$

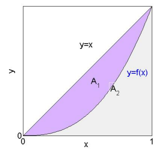
(i) zeros of $f(x) - g(x)$

(ii) the sign of $f(x) - g(x)$



- Gini Index(L.30,Ch.14-1): $G = A_1/A_2 = 2\int_0^1 (x - f(x))dx$





- Integration by parts(L.31,Ch.14-3):

$$\int_a^b uv'dx = uv\Big|_a^b - \int_a^b vu'dx.$$

Problems and Solutions

Example

Evaluate the following indefinite or definite integrals:

(a) $\int_0^1 x^4\sqrt{x^5+1}dx$

(b) $\int xe^{-x^2}dx$

(c) $\int x^2(3x+2)^{-1/3}dx$

(d) $\int_0^1 x^2e^xdx$

(e) $\int_1^2 x^4\ln x dx$

(f) $\int \frac{\sqrt{\ln x}}{x}dx$

Problems and Solutions

Solution for (a) and (b)

(a) Let $u = x^5 + 1$, then $du = 5x^4dx \Rightarrow x^4dx = du/5$:

$$\int x^4\sqrt{x^5+1}dx = \frac{1}{5}\int u^{1/2}du = \frac{2}{15}u^{3/2} = \frac{2}{15}(x^5+1)^{3/2}+C.$$

Therefore,

$$\int_0^1 x^4\sqrt{x^5+1}dx = \frac{2}{15}(x^5+1)^{3/2}\Big|_0^1 = \frac{2(2\sqrt{2}-1)}{15}.$$

(b) Let $u = -x^2$, then $du = -2xdx \Rightarrow xdx = -du/2$:

$$\int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2} + C.$$

Problems and Solutions

Solution for (c)

(c) Let $u = 3x + 2$, then $x = (u - 2)/3$ and $dx = du/3$:

$$\begin{aligned}&\int x^2(3x+2)^{-1/3}dx\\ &= \frac{1}{27}\int (u-2)^2 u^{-1/3}du = \frac{1}{27}\int (u^2-4u+4)^2 u^{-1/3}du\\ &= \frac{1}{27}\int u^{5/3}du - \frac{4}{27}\int u^{2/3} + \frac{4}{27}\int u^{-1/3}du\\ &= \frac{u^{8/3}}{72} - \frac{4u^{5/3}}{45} + \frac{2u^{2/3}}{9} + C\\ &= \frac{(3x+2)^{8/3}}{72} - \frac{4(3x+2)^{5/3}}{45} + \frac{2(3x+2)^{2/3}}{9} + C\\ &= \frac{(3x+2)^{2/3}}{40}(5x^2-4x+4) + C.\end{aligned}$$

Problems and Solutions

Solution for (d)

(d) Let $u = x^2$ and $v' = e^x$, we have $u' = 2x$ and $v = \int e^x dx = e^x$. Hence by integrating by parts, we have

$$\int_0^1 x^2 e^x dx = x^2 e^x\Big|_0^1 - \underbrace{\int 2xe^x dx}_{\text{by part again}}.$$

Then we need to choose $u = 2x$ and $v = e^x$ to obtain

$$\int 2xe^x dx = 2xe^x\Big|_0^1 - 2\int e^x dx = 2e - 2e^x\Big|_0^1 = 2.$$

Finally we have

$$\int_0^1 x^2 e^x dx = x^2 e^x\Big|_0^1 - 2 = e - 2.$$

Problems and Solutions

Solution for (e)

(e) Let $u = \ln x$ and $v' = x^4$, we have $u' = 1/x$ and $v = \int x^4 dx = x^5/5$. Hence by integrating by parts, we have

$$\begin{aligned}\int_1^2 x^4 \ln x dx &= \frac{x^5 \ln x}{5}\Big|_1^2 - \frac{1}{5}\int_1^2 x^4 dx\\ &= \frac{32\ln 2}{5} - \frac{x^5}{20}\Big|_1^2 = \frac{32\ln 2}{5} - \frac{31}{20}.\end{aligned}$$

(f) Let $u = \ln x$, then $du = \frac{dx}{x}$. Hence

$$\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2u^{3/2}}{3} = \frac{2(\ln x)^{3/2}}{3}.$$

Problems and Solutions

Example

Given

$$f(x) = (\ln x)^2 - \ln x,\ \text{with } x > 0$$

(a) find the intervals where $f(x)$ is positive / negative / increasing / decreasing / concave-upwards / concave-downwards

(b) sketch $f(x)$ for $x \in [0.2, 5]$

(c) find the area bounded by $f(x)$ and the x-axis

Solution - Part 1

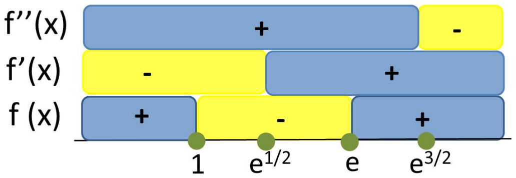
(a) To determine the sign of $f(x)$, we need its zeros:

$$(\ln x)^2 - \ln x = 0 \Rightarrow \ln x = 0, 1 \Rightarrow x = 1, e.$$

$$f'(x) = \frac{2\ln x - 1}{x} \Rightarrow \text{ critical point: } x = \sqrt{e}.$$

$$f''(x) = \frac{3 - 2\ln x}{x^2} \Rightarrow \text{ inflection point: } x = \sqrt{e^3}.$$

Then one can generate the following sign chart:

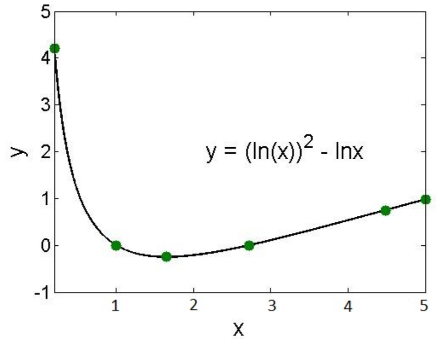


Problems and Solutions

Solution - Part 2

(b) Evaluation of $f(x)$ at characterising points:

x	$f(x)$
0.2	4.1997
1	0
$\sqrt{e}$	-0.25
e	0
$\sqrt{e^3}$	0.75
5	0.9809



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Problems and Solutions

Solution - Part 3

(c) The bounded area then is $S = -\int_1^e ((\ln x)^2 - \ln x)dx$.

To evaluate $\int_1^e (\ln x)^2 dx$, we need $u = (\ln x)^2$ and $v' = 1$, so that $u' = 2\ln x/x$ and $v = \int 1dx = x$. Integration by parts gives

$$\int_1^e (\ln x)^2 dx = x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x dx$$

To evaluate $\int_1^e \ln x dx$, we need $u = \ln x$ and $v' = 1$, so that $u' = 1/x$ and $v = x$. Then

$$\int_1^e \ln x dx = x\ln x\Big|_1^e - \int_1^e 1dx = 1.$$

Therefore, the area is $S = -(\underbrace{e - 2\cdot 1}_{\int_1^e (\ln x)^2 dx} - \underbrace{1}_{\int_1^e \ln x dx}) = 3 - e$.

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