MATH 1003 Review: Part 1. Interest Rates

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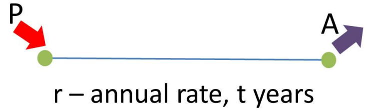
Mathematics of Finance

(a) Simple interest rate (L.1,Ch3.1)

$$A = P(1 + rt)$$

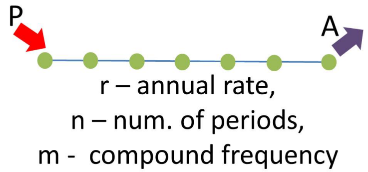


(b) Compound interest rate (L.2,Ch3.2)

$$A = P\left(1 + \frac{r}{m}\right)^{n}$$

- APY - the true interest rate

$$APY = \left(1 + \frac{r}{m}\right)^{m} - 1$$



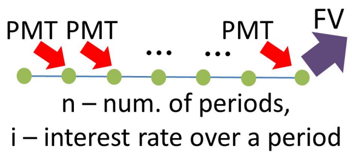
- Continuous interest rate

$$A = Pe^{rt}$$

Mathematics of Finance

(c) (Ordinary) annuity (L.3,Ch3.3)

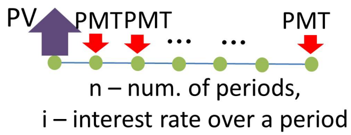
$$FV = PMT \times \frac{(1+i)^n - 1}{i}$$



▷ if FV unknown, annuity
▷ if PMT unknown, sinking fund

(d) Present Value of an Annuity (L.4,Ch3.4)

$$PV = PMT \times \frac{1 - (1+i)^{-n}}{i}$$



▷ if PMT unknown, amortisation
▷ mortgage, n - length of the mortgage contract
▷ Interest earned
= total withdrawal - total payment.

Problems and Solutions

Example

Identify the following events by indicating their associated formulas

- James puts \$ 1000 into a bank, and plans to withdraw it for his graduation travel.
- Peter decides to deposit some money from his scholarship every half a year into an account for his graduation travel.
- For her graduate travel, Laura takes the following plan: work hard to get a good job, so that she may borrow some money for the travel and clear the loan monthly from her future salary.
- Mr and Mrs. Lee will regularly (every three months) save same amount of money to a bank account for their 10-year wedding anniversary ...
- ...

Problems and Solutions

Example

Ann is planning for her retirement after 20 years. She plans to enter a pension scheme with interest rate 5% compounded monthly for 20 years. Her aim is to withdraw \$ 1500 monthly for the next 30 years after her retirement.

(a) How much should she have in the account as her retirement starts to achieve her goal for retirement?

(b) How much is the monthly payment for the pension scheme?

(c) How much interest she will receive after 50 years?

Problems and Solutions

Solution

(a) Amount in the account when she retires (PV of an annuity):

$$PV = 1500\frac{1-(1+0.05/12)^{-30\times 12}}{0.05/12} = 279,422.43$$

(b) Monthly payment for the pension (annuity):

$$PMT = 279422.43\frac{0.05/12}{(1+0.05/12)^{(20\times 12)}-1} = 679.80$$

(c) Total interest received:

$$1500\times 30\times 12 - 679.80\times 20\times 12 = 376,848.00$$

Problems and Solutions

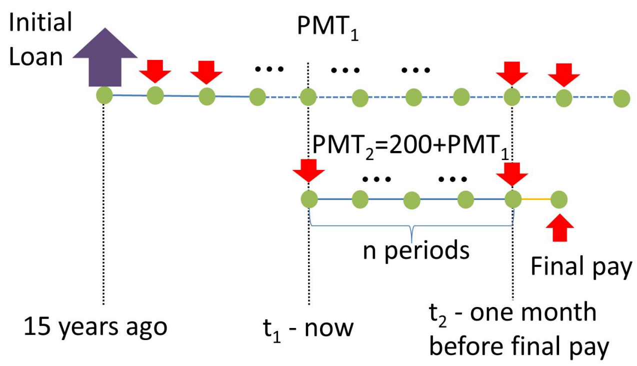
Example

A couple bought a house for 400,000 dollars 15 years ago, she agreed to pay 25% down and sign a 30-year mortgage at 13% compounded monthly. Now the interest rate has dropped but their wages appreciate. So they manage to negotiate a new mortgage contract with annual rate 10% from now on, and the monthly payment will be \$ 200 more than that in the previous mortgage contract.

- How much is the unpaid balance as the new contract starts (now)?
- How long will the new mortgage contract be?
- How much will the final payment be?

Problems and Solutions

Illustration



Keys: unpaid balance at t_1 and t_2 (Mini-example in L.4)

▷ Unpaid balance = PV of an annuity for unpaid periods long

▷ = Initial loan compounded - an annuity for unpaid periods long

Solution - Part 1

(a) PMT_1 for the previous contract

$$PMT_1 = (400000 \times 0.75)\frac{0.13/12}{1-(1+0.13/12)^{-30\times 12}} = 3318.60$$

(b) Unpaid balance sheet at t_1 (now) denoted by PV_0:

$$PV_0 = PMT_1\frac{1-(1+0.13/12)^{-15\times 12}}{0.13/12} = 262289.71$$

(c) Length of the new contract m:

$$PV_0 = PMT_2\frac{1-(1+0.1/12)^{-m}}{0.1/12} \Rightarrow$$

$$m = -\log_{(1+0.1/12)}\left(1-\frac{262289.71\times 0.1/12}{3318.60+200}\right) = 116.9$$

Solution - Part 2

(d) Since m is not an integer, an extra month is needed. They will pay PMT_2 for $n = 116$ months and at the 117-th month from now, they clear the unpaid balance then. The unpaid balance after 116 months is calculated by

$$\underbrace{PV_0 \times \left(1+\frac{0.1}{12}\right)^{116}}_{PV_0 \text{ compounded 116 months}} - \underbrace{PMT_2\frac{(1+0.1/12)^{116}-1}{0.1/12}}_{\text{Annuity for 116 months}} = 3398.20.$$

(e) This value is compounded for another month when the clearance is made:

$$\text{final payment} = 3398.20 \times (1+0.1/12) = 3462.52.$$