MATH 1003 Calculus and Linear Algebra (Lecture 2)

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Notice

1. Problem sets 1 and 2, Web of Work (compulsory) are open.
2. Midterm 1 (25%), 1.5-hour - Date: 8 October (Saturday)
3. Midterm 2 (25%), 1.5-hour - Date: 12 November (Saturday)

Compound Interest

Example

If \$1,000 is deposited at annual interest rate 10% and the bank provides interest

(a) annually;
(b) semiannually;
(c) quarterly;
(d) monthly.

What is the amount of money in the bank after 4 years?

Solution for (a)

- Amount of money in the bank at the end of the 1st year:

$$1000(1+0.1)$$

- Amount of money in the bank at the end of the 2nd year:

$$1000(1+0.1)(1+0.1) = 1000(1+0.1)^2$$

- Amount of money in the bank at the end of the 3rd year:

$$1000(1+0.1)^2(1+0.1) = 1000(1+0.1)^3$$

- Amount of money in the bank at the end of the 4th year:

$$1000(1+0.1)^3(1+0.1) = 1000(1+0.1)^4 = \$1464.1$$

Solution for (b)

The computation is similar to the solution for (a). But now the period of providing interest is half year. Therefore, we have the following:

- Amount of money in the bank at the end of the 1st half-year:

$$1000(1+\frac{0.1}{2})$$

- Amount of money in the bank at the end of the 2nd half-year:

$$1000(1+\frac{0.1}{2})(1+\frac{0.1}{2}) = 1000(1+\frac{0.1}{2})^2$$

- ......
- Amount of money in the bank at the end of the 8th half-year:

$$1000(1+\frac{0.1}{2})^8 = \$1477.5$$

Solutions for (c) and (d)

By similar computation, we obtain the following results:

- For (c), the amount of money in the bank at the end of the 4th year is

$$1000(1+\frac{0.1}{4})^{16} = \$1484.5$$

- For (d), the amount of money in the bank at the end of the 4th year is

$$1000(1+\frac{0.1}{12})^{48} = \$1489.4$$

Compound Interest

The previous example is

Theorem

Let the annual interest rate be r. Let P be the principal (present value). If the bank provides interest m times per year, then after t years, the amount (future value), A, is given by

$$A = P\left(1+\frac{r}{m}\right)^{mt}.$$

The total interest earned is

$$I = P\left(1+\frac{r}{m}\right)^{mt} - P.$$

Check that $I > Prt$.

Compound Interest

Observation

Given a fixed annual interest rate, the more times compounded in a certain period, the more profit a certain amount deposit can make

Example

Period	3 months	6 months	1 year
Interest rate	2.4 p.a.	2.5 p.a.	2.7 p.a.

The annual interest rate generally varies for different periods.

Example 1 Finding Present Value

Example

How much should you invest now at 10% compounded quarterly to have $8,000 toward the purchase of a car in 5 years?

Example 1 Finding Present Value

Solution

Let P be the amount of investment. Then we have

$$P(1+\frac{0.1}{4})^{20}=8000$$

$$\Rightarrow P=\$4882.2$$

A remark: This can be viewed as the present value of $8,000 after 5 years.

Example 2 Computing Growth Time

Example

How long will it take $10,000 to grow to $12,000 if it is invested at 9% compounded monthly?

Example 2 Computing Growth Time

Solution part 1

Let n be the number of months needed for $10,000 to grow to $12,000. Then we have

$$12000=10000(1+\frac{0.09}{12})^n$$

$$\Rightarrow 1.2=(1.0075)^n$$

Example 2 Computing Growth Time

Solution part 2

To solve an equation with an unknown in the power, we need to use the "logarithm":

$$\ln 1.2 = \ln(1.0075)^n$$

$$\ln 1.2 = n \ln(1.0075)$$

$$\Rightarrow n = \frac{\ln 1.2}{\ln 1.0075} = 24.4$$

Therefore, it will take 25 months for $10,000 to grow to $12,000.

Example 3 Finding Interest (Inflation) Rate (Self-Study)

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Solution

Let r be the annual nominal rate of the mutual fund. Then we have

$$128000 = 10000\,(1 + r)^{20}$$

$$\sqrt[10]{12.8} = 1 + r$$

$$\Rightarrow r = 0.29 = 29\%$$

Compound Interest

Recall that

Theorem

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Remark

- $m = 2$: compounded semiannually;
- $m = 4$: compounded quarterly;
- $m = 12$: compounded monthly;

Compound Interest

Remark

$$A = P\left(1 + \frac{r}{m}\right)^{mt}.$$

- $m = 365$: compounded daily;
- $m = 365 * 24 * 60 = 525,600$: compounded every minute;
- $m = 5,256,000$: compounded every 10th of a minute;
- $m \to \infty$, i.e., m gets larger and larger, then what?

If $m \to \infty$, interest is called compounded continuously, we have

Continuous Compound Interest

Definition

As the number m of compounding periods per year increases without bound, the compounded amount approaches a limiting value. This value is given by the following formula:

$$\lim_{m\to\infty} P\left(1 + \frac{r}{m}\right)^{mt} = Pe^{rt}$$
$$A = Pe^{rt},$$

where A is the compounded amount and $e \approx 2.71828$.

Continuous Compound Interest

Example

What amount will an account have after 10 years if \$1500 is invested at an annual rate of 6.75% compounded continuously?

Solution

$$A = 1500e^{0.0675\times 10} = \$2946.05.$$

Remark

This amount is only 18 cents more than the amount you receive by daily compounding.

Annual Percentage Yield

Definition

If a principal is invested at the annual rate r compounded m times a year, then the amount after 1 years is $A = P(1+\frac{r}{m})^m$. The simple interest rate that will produce the same amount A in 1 year is called the annual percentage yield (APY).

Theorem

Formula for APY:

$$APY = (1+\frac{r}{m})^m - 1$$

The APY is also referred to as the effective rate or the true interest rate.

remark

r and m are both determined by a financial institution.

Example 4 Comparison of Different Investments

APY is useful when you want to compare different investment/loan schemes.

Example

Three banks offer 1-year certificates of deposit (CD):

(a) Lion bank pays 3.97% compounded daily
(b) Chatter bank pays 3.95% compounded monthly
(c) Asian bank pays 3.98% compounded quarterly

Find APY for each of these banks and determine which bank offers the greatest return.

Example 4 Comparison of Different Investments

Solution

(a) APY for Lion bank is

$$\left(1+\frac{0.0397}{360}\right)^{360} - 1 = 4.05\%$$

(b) APY for Chatter bank is

$$\left(1+\frac{0.0395}{12}\right)^{12} - 1 = 4.02\%$$

(c) APY for Asian bank is

$$\left(1+\frac{0.0398}{4}\right)^{4} - 1 = 4.04\%$$

Therefore, the CD of Lion bank has the greatest return.

Example 5 Find the Annual Nominal Rate (Optional)

Example

A savings and loan wants to offer a CD with a monthly compounding rate that has an APY of 7.2%. What annual nominal rate compounded monthly should they use?

Solution

Let r be the annual nominal rate. Then we have

$$0.072 = \left(1 + \frac{r}{12}\right)^{12} - 1$$

$$\Rightarrow r = 0.0697 = 6.97\%$$