MATH 1003 Calculus and Linear Algebra (Lecture 3)

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Future Value of an Annuity

Definition

An annuity is a sequence of equal periodic payments. We call it an ordinary annuity if the payments are made at the end if each time interval. The amount, or future value, of an annuity is the sum of all payment plus all interest earned.

Remark

In this course, all annuities are assumed to be ordinary.

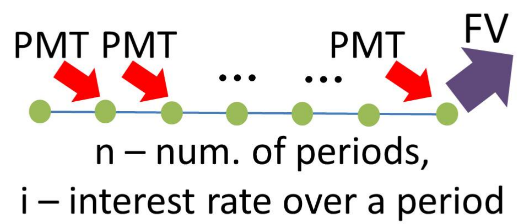
An Example

Example

Suppose you deposit \$100 every 6 months into an account that pay 6% compounded semiannually. If you make six deposits, one at the end of each interest payment period, over 3 years, how much money will be in the account after the last deposit is made?

An Example

Idea: We compute the future value of each payment one by one and then sum them up.



- Future value of the 1st payment: $100(1+\frac{0.06}{2})^5$
- Future value of the 2nd payment: $100(1+\frac{0.06}{2})^4$
- Future value of the 3rd payment: $100(1+\frac{0.06}{2})^3$
- ......

An Example

Therefore, the future value of the annuity is

$$FV = 100 + 100(1+\frac{0.06}{2}) + 100(1+\frac{0.06}{2})^2 + \cdots + 100(1+\frac{0.06}{2})^5$$

This type of sum is called the geometric sum. It can be conveniently computed by the following formula:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}$$

Now, let $a = 100$ and $r = 1 + \frac{0.06}{2}$. Then by the formula, we have

$$FV = 100 \cdot \frac{(1+\frac{0.06}{2})^6 - 1}{\frac{0.06}{2}} = \$646.84$$

Formula for the Future Value of an Annuity

We now introduce the formula of the future value of an annuity in the term of the notations used in finance.

Theorem

Let FV = future value, PMT = periodic payment, i = interest rate per period and n = number of payments. We have

$$FV = PMT\frac{(1+i)^n - 1}{i}.$$

Thus the total interest of an annuity is

$$Interest = FV - n \times PMT.$$

Example 1

Example

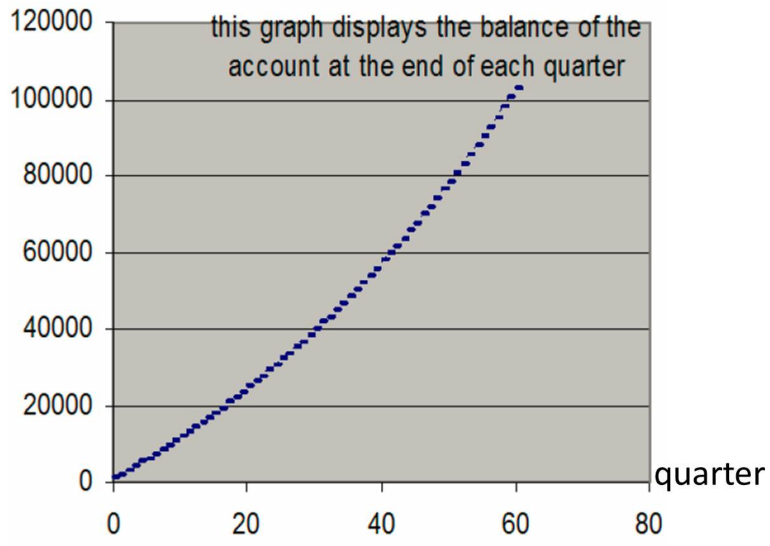
What is the future value of an annuity at the end of 15 years if \$1,000 is deposited each quarter (3 months) into an account earning an interest rate of 6.5% ? How much of this value is interest?

Solution

$$FV = 1000 \cdot \frac{(1+0.065/4)^{4*15} - 1}{0.065/4} = \$100336.68$$

Interest earned $= 100336.68 - 60 \times 1000 = \$40336.68$

Information from Graphics



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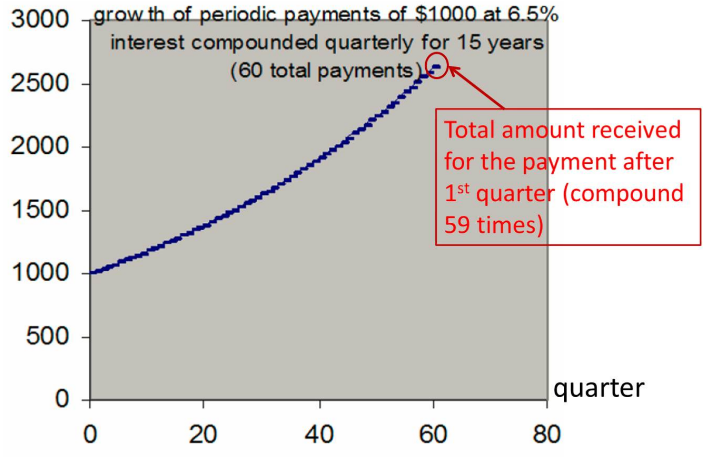

How long will it take for a PMT to get doubled?

Sinking Fund

Any account that is established for accumulating funds to meet future obligations or debts is called a sinking fund.

Example

Suppose the parents of a newborn child decide that on each of the child's birthdays up to the 17^{th} year, they will deposit \$PMT in an account that pays 6% compounded annually. The money is to be used for the college expenses. What should the annual deposit \$PMT be in order for the amount in the account to be \$80,000 after 17^{th} deposit?

Sinking Fund

Solution

Since the parents hope that the future value of all the periodic deposits they will made, i.e. the annuity, can cover the child's college expenses, then we should set $FV = \$80000$, $n = 17$ and $i = 0.06$. Hence we obtain

$$80000 = PMT\frac{(1+0.06)^{17}-1}{0.06}$$

Solving for PMT, we get $PMT = \$2835.58$.

Formula for Sinking Fund Payment

Suppose the payments of a sinking fund are to be made in the form of an ordinary annuity, then we obtain the formula for the sinking fund payment:

$$PMT = FV\frac{i}{(1+i)^n - 1}.$$

Example 2 Combination of Financial Tools

Example

Lion bank offered a money market account with an APY of 5%.

(a) If interest is compounded monthly, what is the annual interest rate?

(b) If a company wishes to have $1,000,000 in this account after 8 years, what equal deposit should be made each month?

Example 2 Combination of Financial Tools

Solution

(a) Given the APY, we need to find the annual interest rate r:

$$(1+\frac{r}{12})^{12}-1=0.05$$

$$\Rightarrow r = 4.89\%$$

(b) $FV = 1000000$ and PMT is the unknown period payment amount. Then we have

$$PMT = 1000000 \cdot \frac{\frac{0.0489}{12}}{(1+\frac{0.0489}{12})^{96}-1}$$

$$\Rightarrow PMT = \$8532.62$$

Example 3 Combination of Financial Tools

Example

Peter deposits $1000 monthly into MPF (a retirement plan) that earn 3% compounded monthly. Due to a change in employment, these deposits stop after 10 years, but the account continues to earn interest until Peter retires 25 years after the last deposit was made. How much is in the account when Peter retires?

Example 3 Combination of Financial Tools

Solution

First, we need to calculate the future value of Peter's MPF after 10 years:

$$FV = 1000 \cdot \frac{(1+\frac{0.03}{12})^{10\times 12}-1}{\frac{0.03}{12}}$$

$$\Rightarrow FV = \$139741.4189$$

Then, in the next 25 years, no deposit was made. Therefore, the total amount of money Peter eventually has can be calculated by the straightforward compound interest formula:

$$\text{Total amount} = 139741.4189(1+\frac{0.03}{12})^{25\times 12} = \$295555.83$$

This is compared to two time lines in example 4 of Lecture 1.