MATH 1003 Calculus and Linear Algebra (Lecture 4)

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Reminder

Midterm Exam

- Time: 7 Oct. 2017 (Sat), 10:30–12:00,
- Location: TBA.

Calculators

- Calculators with functions such as $1.23^{26.7}$, $\log_2 2568.9$ will be needed in the midterm exam,
- but only calculators with no matrix-operation will be allowed.
- The law will be enforced strictly.

Present Value of an Annuity

Example

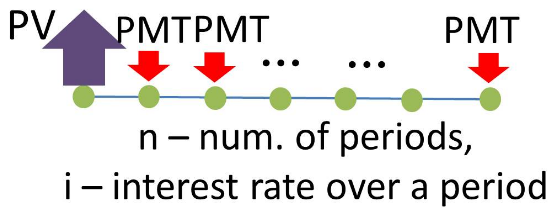
How much should you deposit in a bank account paying 6% compounded semi-annually in order to be able to withdraw \$100 every 6 months for the next 3 years? (After the last withdrawal is made, no money is to be left in the account.)

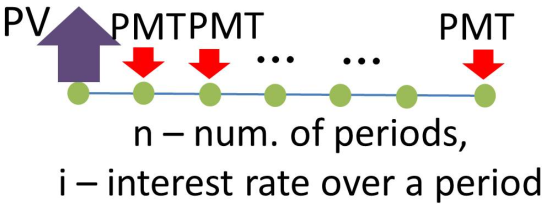
Remark

- Obviously, you do not need to deposit \$600 because part of your future withdrawals can be covered by the interest provided.
- From the bank's viewpoint, the initial deposit can be regarded as the present value of the annuity.

Present Value of an Annuity

Idea: We compute the present value of each withdrawal one by one and sum them up.





- Present value of the 1st withdrawal: $100(1+\frac{0.06}{2})^{-1}$
- Present value of the 2nd withdrawal: $100(1+\frac{0.06}{2})^{-2}$
- Present value of the 3rd withdrawal: $100(1+\frac{0.06}{2})^{-3}$
- ......
- Present value of the 6th withdrawal: $100(1+\frac{0.06}{2})^{-6}$

Present Value of an Annuity

Therefore, the amount of inital deposit is

$$
\begin{aligned}
&100(1+\frac{0.06}{2})^{-1}+100(1+\frac{0.06}{2})^{-2}+\cdots+100(1+\frac{0.06}{2})^{-6}\\
&=(1+\frac{0.06}{2})^{-6}\left(100+100(1+\frac{0.06}{2})+\cdots+100(1+\frac{0.06}{2})^{5}\right)\\
&=(1+\frac{0.06}{2})^{-6}\left(100\cdot\frac{(1+\frac{0.06}{2})^{6}-1}{\frac{0.06}{2}}\right)\\
&=100\cdot\frac{1-(1+\frac{0.06}{2})^{-6}}{\frac{0.06}{2}}=\$541.72
\end{aligned}
$$

Formula for the Present Value of an Annuity

Theorem

Let PV = present value of all payments(withdrawals), PMT = periodic payment(withdrawal), i = interest rate per period and n = number of payments(withdrawals). We have

$$
\begin{aligned}
PV &= PMT(1+i)^{-n}+PMT(1+i)^{-(n-1)}+\cdots+PMT(1+i)^{-1}\\
&= PMT\frac{1-(1+i)^{-n}}{i}.
\end{aligned}
$$

Example 1 A Taste of the Formula

Example

Recently Lion bank offered an ordinary annuity that earned 5% compounded annually. How much should a person pay into his account, so that he/she can make 20 equal annual withdrawals of \$25000, reducing the balance in the account to zero for the next 20 years?

Example 1 To be continued

Solution

Let $PMT = 25000$, $n = 20$, $i = 0.05$ and PV be the required amount of money. Then we have

$$PV = 25000 \cdot \frac{1-(1+0.05)^{-20}}{0.05}$$

$$\Rightarrow PV = \$311555.2586$$

Example

To make for that amount ($311555.2586), he/she plans to make equal annual deposits into this account for 30 years. How much must be deposited annually?

Example 1 To be continued

Solution

Let PMT be the annual deposit needed to accumulate $311555.2586. We have

$$\$311555.2586 = PMT \cdot \frac{(1+0.05)^{30}-1}{0.05}$$

$$\Rightarrow PMT = \$4689.35$$

Example 1 Combination of the Two Problems

Example

Recently Lion bank offered an ordinary annuity that earned 5% compounded annually. A person plans to make equal annual deposits into this account for 30 years, so that he/she can make 20 equal annual withdrawals of $25000, reducing the balance in the account to zero for the next 20 years. How much must be deposited annually to accumulate sufficient funds to provide for these payments? How much total interest is earned during this entire 50-year process?

Example 1 Combination of the Two Problems

To compute the total interest earned in the entire 50-year process, we have

$$\begin{aligned}\text{Interest earned} &= \text{Total withdrawal} - \text{Total payment}\\ &= 25000 \times 20 - 4689.35 \times 30 = \$359319.39\end{aligned}$$

Amortization

The formula for the present value of an annuity can also be used in the following situation:

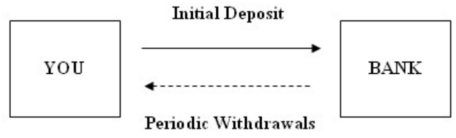
Example

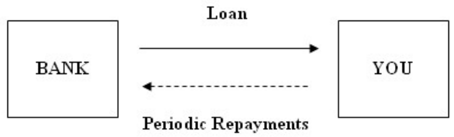
Suppose you borrow $10000 from a bank to buy a car and agree to repay the loan in 24 equal monthly payments, including all interest due. If the bank charges 1% per month on the unpaid balance i.e., 12% per year compounded monthly, how much should each payment be to retire the total debt, including interest in 24 months?

This is called an amortization.

Amortization

The following two kinds of transactions between you and the bank are basically the same:





Solution to the Previous Example

In this example, $PV = 10000$ is the total loan and PMT is the amount of the monthly payment. Then we have

$$10000 = PMT \cdot \frac{1-(1+0.01)^{-24}}{0.01}$$

$$\Rightarrow PMT = \$470.73$$

Amortization Formula

In general, amortizing a debt means that the debt is retired in a given length of time by equal periodic payments that include compound interest. Paying mortgage is an example. We have the following amortization formula for PMT:

$$PMT = PV \frac{i}{1-(1+i)^{-n}}$$

Example 2 Mortgage

Example

A family purchased a home 10 years ago for \$3000000. The home was financed by paying 30% down and signing a 25-year mortgage at 6% on the unpaid balance. The net market value of the house (amount received after subtracting all costs involved in selling the house) is now \$4500000, and the family wishes to sell the house. How much equity does the family have in the house now after making 120 monthly payments?
(note: equity = current net market value - unpaid loan balance)

Example 2 Mortgage

Solution Part 1

The total loan:

$$3000000 \times 0.7 = \$2100000$$

Let PMT be the monthly payment that the family made. Then we have

$$PMT = 2100000 \cdot \frac{\frac{0.06}{12}}{1-(1+\frac{0.06}{12})^{-25\times 12}}$$

$$\Rightarrow PMT = \$13530.3294$$

Now, we would like to calculate the unpaid loan balance of the family after making 120 monthly payments.

Example 2 Mortgage

Solution Part 2

The unpaid loan balance is

$$13530.3294 \times \frac{1-(1+\frac{0.06}{12})^{-180}}{\frac{0.06}{12}} = \$1603391.59$$

or

$$2100000\times(1+\frac{0.06}{12})^{120} - 13530.3294\times\frac{(1+\frac{0.06}{12})^{120}-1}{\frac{0.06}{12}} = \$1603391.59$$

Since the equity = current net market value - unpaid loan balance, we have

$$\text{Equity} = 4500000 - 1603391.59 = \$2896608.41.$$

Why not PMT $\times 180$

A mini example

Borrow \$ 100, annual rate on unpaid balance is 2%, pay back annually for 4 years. The unpaid balance at the end of year 2?

The annual payment (Amortization) will be

$$PMT = 100 \times \frac{0.02}{1-1.02^{-4}} = 26.26.$$

	Year 0	Year 1	Year 2	Year 3	Year 4
Principal					
Interest incurred					
Payment					
Payment to principal					
Unpaid balance					

Why not PMT $\times 180$

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	Year 0	Year 1	Year 2	Year 3	Year 4
Principal	100	75.74	50.995	25.755	0.01
Interest incurred	0	2	1.515	1.02	0.515
Payment	0	26.26	26.26	26.26	26.26
Payment to principal	0	24.26	24.745	25.24	25.745
Unpaid balance	0	75.74	50.995	25.755	0.01

Why not PMT $\times 180$

Unpaid balance after the first m periods out of total M periods =

- compound interest for m periods - an annuity of m payments

$$100 \times 1.02^m - 26.26 \times \frac{1.02^m - 1}{0.02};$$

- present value of an annuity for $(M - m)$ periods

$$26.26 \times \frac{1 - 1.02^{-(M-m)}}{0.02}.$$

Remark

Same amount is different at different time!! When summed up, all payments/loans should be of the same time by discounting/cournting interests.

Review

- Number of stories (choose formulas)
- Draw the time line, if there are more than 1 stories, find the bridge variables (Simultaneous event!!)
- Implementation of the formulas