MATH 1003 Calculus and Linear Algebra (Lecture 5)

Maosheng Xiong
Department of Mathematics, HKUST

Systems of Linear Equations in Two Variables

The following is an example of systems of linear equations in two variables.

Example

Given the following system of linear equations:

$$\begin{cases} x - 2y &= 2 \\ x + y &= 5 \end{cases}$$

Find its solution.

Remark

- A collection of equations is called a system.
- There are two unknowns (or variables) x and y in the system.
- To find a solution to the system, we need to find the values of x and y that satsify the two equations in the system.

Method of Substitution

For the system

$$\begin{aligned} x - 2y &= 2 \qquad (1) \\ x + y &= 5 \qquad (2) \end{aligned}$$

Rewrite (1) as $x = 2 + 2y$ (3) and substitute it into (2):

$$(2 + 2y) + y = 5$$

Solving, we get $y = 1$.

Substitute the value of y into (3), we have $x = 2 + 2 \times 1 = 4$.

Hence the solution is $x = 4, y = 1$.

Method of Elimination

Multiply both sides of (2) by 2, we have

$$2x + 2y = 10 \qquad (4)$$

Add (1) and (4) together, we have

$$(x - 2y) + (2x + 2y) = 2 + 10$$

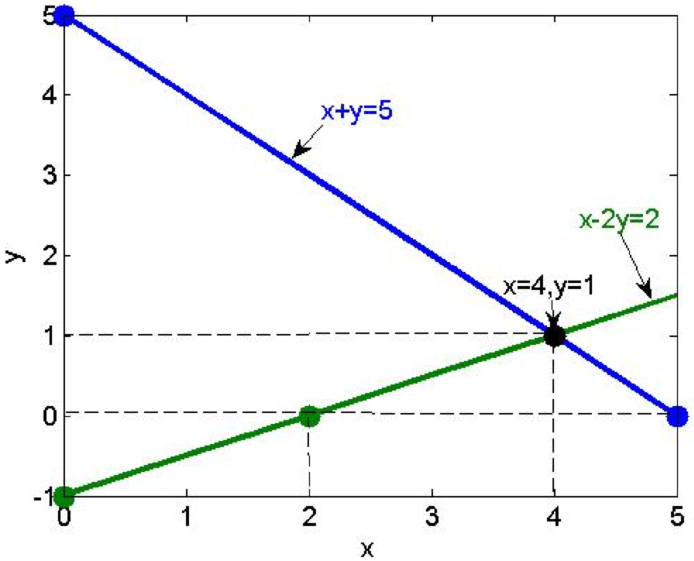
$$\Rightarrow 3x = 12$$

Solving, we have $x = 4$.

Substitute the value of x into (1), we have $4 - 2y = 2$. Solving, we have $y = 1$.

Hence the solution is $x = 4, y = 1$.

Graphical Method



General Linear Systems

Definition

Given the linear system

$$\begin{cases} ax + by &= h \\ cx + dy &= k \end{cases}$$

where a, b, c, d, h and k are constants. A pair of number (x_0, y_0) is a solution of this system if each equation is satisfied by this pair.

Question: Can a linear system always be solved? Can a linear system have more than one solution? (Hint: Imagine how many ways can two straight lines intersect each other.)

Examples

Example

Solve each of the following systems

(a) $\begin{cases} x + y &= 4 \\ 2x - y &= 2 \end{cases}$

(b) $\begin{cases} 6x - 3y &= 9 \\ 2x - y &= 3 \end{cases}$

(c) $\begin{cases} 2x - y &= 4 \\ 6x - 3y &= -18 \end{cases}$

Solution of (a)

$$\begin{cases} x + y &= 4 \\ 2x - y &= 2 \end{cases}$$

Add two equations together, we have

$$(x + y) + (2x - y) = 4 + 2$$

$$\Rightarrow 3x = 6$$

Therefore, $x = 2$ and by substituting it into the first equation, we get $y = 2$.

There is only one solution: $x = 2, y = 2$.

Solution of (b)

$$\begin{cases} 6x - 3y &= 9 \\ 2x - y &= 3 \end{cases}$$

Multiply the second equation by -3 and add to the first equation, we have

$$(6x - 3y) + (-6x + 3y) = 9 - 9$$
$$\Rightarrow 0 = 0$$

Notice that this happens because two equations are essentially the same. Therefore, (x, y) is a solution to the system if it satifies any one equation.

There are infinitely many solutions.

Solution of (c)

$$\begin{cases} 2x - y &= 4 \\ 6x - 3y &= -18 \end{cases}$$

Multiply the first equation by -3 and add to the second equation, we have

$$(-6x + 3y) + (6x - 3y) = -12 - 18$$
$$\Rightarrow 0 = -30$$

A contradiction is obtained. Therefore, there does not exist (x, y) that satisfies the system.

There is no solution.

Types of Linear Systems

Theorem

The linear system

$$\begin{cases} ax + by &= h \\ cx + dy &= k \end{cases} \tag{3}$$

must have

(a) *Exactly one solution. In this case, we call the system (3) independent consistent.*

(b) *Infinitely many solutions. In this case, we call the system (3) dependent consistent.*

(c) *No solution. In this case, we call the system (3) inconsistent.*

Supply and Demand

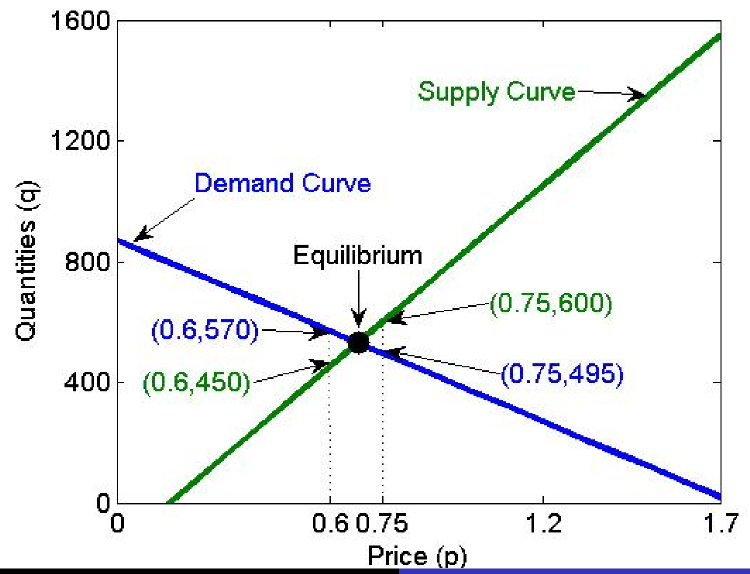
Example

At \$0.6 per bottle, the daily supply for milk is 450 bottles, and the daily demand is 570 bottles. When the prices is raised to \$0.75 per bottle, the daily supply increases to 600 bottles, and the daily demand decreases to 495 bottles. Assume that the supply and the demand equation are linear. Find the equilibrium.

Remark

- The equilibrium is the price and quantity that satify both the demand and supply equations.
- The example is not quite realistic because the demand and supply equations are seldom linear.

Supply and Demand



Supply and Demand

p Price of bottle	Supple	Demand
0.6	450	570
0.75	600	495

Solution Part 1

Let \$$p$ be the price and q be number of bottles of milk.
Given the information about the demand of milk, we can construct the demand equation as follows:

$$\frac{q-570}{495-570}=\frac{p-0.6}{0.75-0.6}$$

$$\Rightarrow p+0.002q=1.74$$

Supply and Demand

p Price of bottle	Supple	Demand
0.6	450	570
0.75	600	495

Solution Part 1

Similarly, we can construct the supply equation as follows:

$$\frac{q-450}{600-450}=\frac{p-0.6}{0.75-0.6}$$

$$\Rightarrow p-0.001q=0.15$$

Supply and Demand

Solution Part 2

The linear system for this problem is

$$\begin{aligned} p+0.002q &= 1.74 \\ p-0.001q &= 0.15 \end{aligned}$$

Solving, we have $p=0.68, q=530$, which is the required equilibrium.

Homework

1. Check `www.math.ust.hk/~mamsxiong` or **Canvas** for updates.
2. WeBWork 1 and 2. Check the due date.
3. Practice Exercises from the Textbook (Optional).