MATH 1003 Calculus and Linear Algebra (Lecture 6)

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Matrices

Definition

A matrix is a rectangular array of numbers written within brackets.

Example

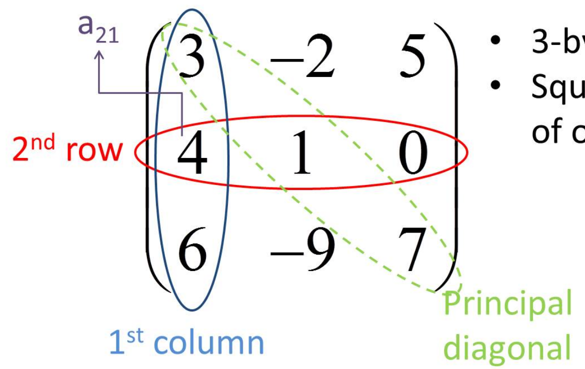
$$A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 1/2 & -\frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 6.8 & 2.3 \\ \frac{1}{2} & \sqrt{2} \end{bmatrix}.$$

Some Definitions

Definition

- Each number in a matrix is called an element of the matrix.
- If a matrix has m rows and n columns, it is called an $\mathbf{m} \times \mathbf{n}$ matrix.
- A matrix with n rows and n columns is called a square matrix of order **n**.
- The position of an element in a matrix is given by the row and the column containing the element. The element on the i^{th} row and j^{th} column is denoted by a_{ij}.
- The principal diagonal of a matrix A consists of the element $a_{11}, a_{22}, a_{33}, \ldots$.

Illustration



- 3-by-3 matrix
- Square matrix of order 3

Where is the smallest number in this matrix?

Linear Systems and Augmented Matrices

Consider the following linear system:

$$\begin{cases} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{cases}$$

We will represent it as an augmented matrix as follows:

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

General Case

In general, associated with each linear system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 &= k_1 \\ a_{21}x_1 + a_{22}x_2 &= k_2 \end{cases}$$

where x_1 and x_2 are variables, is the augmented matrix of the system:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right].$$

Row Operations

When using the method of elimination to solve a linear system, we usually need to multiply an equation by a non-zero constant and then add the equation to another one. Here we want to consider such operations on the corresponding augmented matrix. The following are the three basic types of operations:

- Two rows are interchanged:
  $R_i \leftrightarrow R_j$ means interchanging i^{th} and j^{th} rows.
- A row is multiplied by a non-zero constant:
  $kR_i \to R_i$ means multiplying i^{th} row by a nonzero constant k.
- A constant multiple of one row is added to another row:
  $R_i + kR_j \to R_i$ means that i^{th} row is added by k multiple of j^{th} row.

Notice that the above operations are called the row operations.

An Example

Consider the augmented matrix $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$:

- $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$
- $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \xrightarrow{2R_2 \to R_2} \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & -4 & 14 \end{array}\right]$
- $\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) \xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{cc|c} 5 & 0 & 15 \\ 1 & -2 & 7 \end{array}\right]$

Properties of Row Operations

- All row operations are reversible.
- If A and B are augmented matrices such that one can be transformed to another by a sequence of row operation(s), then A and B are said to be equivalent.
- If two linear systems have equivalent augmented matrices, both systems have the same set of solution(s).

Solving Linear Systems by Row Operations

The aim of using row operations is to reduce the augmented matrix to one of the following simple forms:

(A) $\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$, where m and n are real numbers.

(B) $\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$, where m and n are real numbers.

(C) $\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$, where p, m and n are real numbers and $p \neq 0$.

Conclusion

1. A linear system has a unique solution if its augmented matrix reduces to Form A (independent consistent).
2. A linear system has infinitely many solutions if its augmented matrix reduces to Form B (dependent consistent).
3. A linear system has no solution if its augmented matrix reduces to Form C (inconsistent).

Examples

Example

Solve the following systems of linear equations:

(a) $\begin{cases} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{cases}$

(b) $\begin{cases} 2x_1 - x_2 &= 4 \\ -6x_1 + 3x_2 &= -12 \end{cases}$

(c) $\begin{cases} 2x_1 + 6x_2 &= -3 \\ x_1 + 3x_2 &= 2 \end{cases}$

Solution to (a)

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right] \xrightarrow{R_2+(-3)R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1+2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

The solution is $x_1 = 3, x_2 = -2$. The system is independent consistent.

Solution to (b)

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right] \xrightarrow{R_2+3R_1 \to R_2} \left[\begin{array}{cc|c} 2 & -1 & 4 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

There are infinitely many solutions. The system is dependent consistent.

Solution to (c)

$$\left[\begin{array}{cc|c} 2 & 6 & -3 \\ 1 & 3 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 6 & -3 \end{array}\right] \xrightarrow{R_2+(-2)R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & -7 \end{array}\right]$$

There is no solution. The system is inconsistent.

About Homework

Level of importance

1. Web of work (counted as 10% of the final mark, keep trying until you get it correct before the deadline)
2. Problems to be covered by TA class (semi-compulsory)
3. Problems in the mini-lecture (with answers, optional, can be downloaded from the course webpage)
4. Exercises in the textbook (listed in the course webpage, optional)