MATH 1003 Calculus and Linear Algebra (Lecture 7)

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Gauss-Jordan Elimination

In the last lecture, we learned how to solve a system of 2 equations with 2 variables by performing row operations on its augmented matrix. Now, we generalize this method so that it can be applied to any system of linear equations. It is called the Gauss-Jordan elimination.

The idea is very simple: We transform the corresponding augmented matrix by row operations to a simple form. And the solution(s) to the linear system corresponding to this simple form (which is(are) also the solution(s) to the original system) can be obtained easily.

Reduced Form

Definition

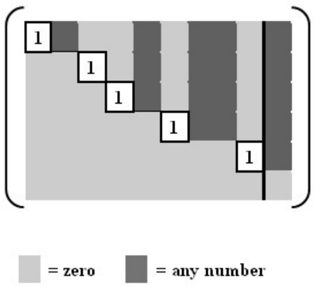
A matrix is said to be in reduced form if it satisfies

1. Each row consisting entirely of zeros is below any row having at least one nonzero element.
2. The leftmost nonzero element in each row is 1.
3. All other elements in the column containing the leftmost 1 of a given row are zeros.
4. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Gauss-Jordan elimination is the process of transforming an augmented matrix to its reduced form.

Reduced Form

A typical reduced form should look like this:



Examples

Example

Indicate which condition in the definition is violated for each matrix. State the row operation required to transform the matrix into reduced form.

(a) $\left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & 0 & 3 \end{array}\right]$ Ans: 4 $(R_1 \leftrightarrow R_2)$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ Ans: 3 $(R_1 + 2R_2 \to R_1)$

(c) $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right]$ Ans: 1 $(R_2 \leftrightarrow R_3)$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$ Ans: 2 $(\frac{1}{2}R_2 \to R_2)$

Example 1

Solving system of linear equations:

Example

Solve $\begin{cases} 3x_1 + 8x_2 - x_3 & = -18 \\ 2x_1 + x_2 + 5x_3 & = 8 \\ 2x_1 + 4x_2 + 2x_3 & = -4 \end{cases}$

Example 1 - Solution Part 1

$$\left[\begin{array}{ccc|c} 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \\ 2 & 4 & 2 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ 2 & 1 & 5 & 8 \\ 3 & 8 & -1 & -18 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 2 & 1 & 5 & 8 \\ 3 & 8 & -1 & -18 \end{array}\right] \xrightarrow[(-3)R_1+R_3 \to R_3]{(-2)R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & -3 & 3 & 12 \\ 0 & 2 & -4 & -12 \end{array}\right]$$

Example 1 - Solution Part 2

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & -4 & -12 \end{array}\right] \xrightarrow[(-2)R_2+R_3 \to R_3]{(-2)R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[(-3)R_3+R_1 \to R_1]{R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example 1 - Solution Part 3

Therefore, from the reduced form, we can easily obtain the solution to the linear system:

$$\begin{cases} x_1 = 0 \\ x_2 = -2 \\ x_3 = 2 \end{cases}$$

Example 2

Example

Solve $\begin{cases} 2x_1 - x_2 - 3x_3 & = 8 \\ x_1 - 2x_2 & = 7 \end{cases}$

Example 2 - Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right]$$

$$\xrightarrow{(-2)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right] \xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

Example 2 - Infinite Number of Solutions

The linear system corresponding to the reduced form is

$$\begin{cases} x_1 \quad\quad - 2x_3 = 3 \\ \quad\quad x_2 - x_3 = -2 \end{cases}$$

Now we classify the variables of the system into two types:

- Non-free variables: Variables corresponding to the columns that contain the leftmost 1's (x_1, x_2)
- Free variables: The remaining variables (x_3)

Example 2 - Infinite Number of Solutions

We set the free variable $x_3 = t$, where t is a real parameter. Then by the system of the reduced form, we have

$$\begin{cases} x_1 \quad - 2t = 3 \\ \quad x_2 - t = -2 \end{cases}$$

Therefore, the solutions are

$$\begin{cases} x_1 = 3 + 2t \\ x_2 = -2 + t \\ x_3 = t \end{cases}$$

where t is any real number.

Example 3

Example

Solve $\begin{cases} 4x_1 - x_2 + 2x_3 &= 3 \\ -4x_1 + x_2 - 3x_3 &= -10 \\ 8x_1 - 2x_2 + 9x_3 &= -1 \end{cases}$

Example 3 Solutions

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ -4 & 1 & -3 & -10 \\ 8 & -2 & 9 & -1 \end{array}\right] \xrightarrow{\frac{1}{4}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ -4 & 1 & -3 & -10 \\ 8 & -2 & 9 & -1 \end{array}\right]$$

$$\xrightarrow[(-8)R_1+R_3 \to R_3]{4R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 5 & -7 \end{array}\right] \xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 5 & -7 \end{array}\right]$$

$$\xrightarrow[(-5)R_2+R_3 \to R_3]{(-\frac{1}{2})R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & 0 & -\frac{17}{4} \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & -42 \end{array}\right]$$

The last row indicates $0 = -42$ a contradiction, the linear system has no solution.

Remarks

The deadline of WeBWork Homework 1 is

at 11:55pm, 15 September (Friday).

The deadline of WeBWork Homework 2 is

at 11:55pm, 22 September (Friday).

The WeBWork Homework 3 is open. You should finish it on 6 October which is before Midterm 1.