MATH 1003 Calculus and Linear Algebra (Lecture 8)

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Example 1 - Tank Car Leases

Example

A chemical manufacturer wants to lease a fleet of 24 railroad tank cars with a combined carrying capacity of 520,000 gallons. Tank cars with three different carrying capacities are available: 8,000 gallons, 16,000 gallons, and 24,000 gallons. How many of each type of tank car should be leased?

Solution to Example 1 Part 1

To formulate the problem, we first need to define relevant quantities as variables:

Let x_1 be the number of tank cars with a capacity of 8,000 gallons
Let x_2 be the number of tank cars with a capacity of 16,000 gallons
Let x_3 be the number of tank cars with a capacity of 24,000 gallons

Given the criteria stated in the problem, we can obtain the following linear system:

$$\begin{cases} x_1 + x_2 + x_3 &= 24 \\ 8000x_1 + 16000x_2 + 24000x_3 &= 520000 \end{cases}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 8000 & 16000 & 24000 & 520000 \end{array}\right]$$

Solution to Example 1 Part 2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 8000 & 16000 & 24000 & 520000 \end{array}\right] \xrightarrow{\frac{1}{8000}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 1 & 2 & 3 & 65 \end{array}\right]$$

$$\xrightarrow{R_2+(-1)R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & 1 & 2 & 41 \end{array}\right] \xrightarrow{R_1+(-1)R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -17 \\ 0 & 1 & 2 & 41 \end{array}\right]$$

The linear system corresponding to the reduced form is

$$\begin{cases} x_1 \quad - \quad x_3 = -17 \\ \quad x_2 + 2x_3 = 41 \end{cases}$$

Solution to Example 1 Part 3

Since x_3 is the free variable, we let $x_3 = t$, where t is any real number. Then the general solution can be written as follows:

$$\begin{cases} x_1 &= -17 + t \\ x_2 &= 41 - 2t \\ x_3 &= t \end{cases}$$

where t is any real number.

However, the definitions of x_1, x_2 and x_3 imply that the solutions must be non-negative integers. Therefore, we need to derive the possible range of values of t such that the general solution make sense for this problem:

$$\begin{cases} -17 + t & \geq 0 \\ 41 - 2t & \geq 0 \\ t & \geq 0 \end{cases}$$

Solution to Example 1 Part 4

Solving the inequalities, we finally obtain all the relevant solutions to the linear system:

$$\begin{cases} x_1 &= -17 + t \\ x_2 &= 41 - 2t \\ x_3 &= t \end{cases}$$

where t is any integer such that $17 \leq t \leq 20$.

Example 2 - Boat Production Description

Example

A small manufacturing plant makes three types of inflatable boats: one-person, two-person, and four-person models. Each boat requires the services of three departments, as listed in the table. The cutting, assembly and packaging departments have available a maximum of 380, 330 and 120 labor-hours per week, respectively.

Department	One-Person Boat	Two-Person Boat	Four-Person Boat
Cutting	0.5hr	1.0hr	1.5hr
Assembly	0.6hr	0.9hr	1.2hr
Packaging	0.2hr	0.3hr	0.5hr

Example 2 - Boat Production Questions

(a) How many boats of each type must be produced each week for the plant to operate at full capacity?

(b) How is the production schedule in part (a) affected if the packaging department is no longer used?

(c) How is the production schedule in part (a) affected if the four-person boat is no longer produced?

Solution to Example 2 Part 1

Let x_1, x_2 and x_3 be the number of one-person boats, two-person boats and four-person boats produced each week respectively.

According to the given table, we have the following linear system:

$$\begin{cases} 0.5x_1 + x_2 + 1.5x_3 &= 380 \\ 0.6x_1 + 0.9x_2 + 1.2x_3 &= 330 \\ 0.2x_1 + 0.3x_2 + 0.5x_3 &= 120 \end{cases}$$

Solution to Example 2 Part 2

$$\left[\begin{array}{ccc|c} 0.5 & 1 & 1.5 & 380 \\ 0.6 & 0.9 & 1.2 & 330 \\ 0.2 & 0.3 & 0.5 & 120 \end{array}\right] \xrightarrow[10/3R_2 \to R_2]{2R_1 \to R_1, 10R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 2 & 3 & 4 & 1100 \\ 2 & 3 & 5 & 1200 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & -1 & -2 & -420 \\ 0 & -1 & -1 & -320 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & 1 & 2 & 420 \\ 0 & -1 & -1 & -320 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -80 \\ 0 & 1 & 2 & 420 \\ 0 & 0 & 1 & 100 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 220 \\ 0 & 0 & 1 & 100 \end{array}\right]$$

Therefore, the solution is $x_1 = 20, x_2 = 220$ and $x_3 = 100$.

Solution to Example 2 Part 3

If the packaging department is no longer used, we should drop the last equation in the linear system i.e. the corresponding augmented matrix should be

$$\left[\begin{array}{ccc|c} 0.5 & 1 & 1.5 & 380 \\ 0.6 & 0.9 & 1.2 & 330 \end{array}\right]$$

Using the steps of Gauss-Jordan elimination in (a), we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -80 \\ 0 & 1 & 2 & 420 \end{array}\right]$$

Part of the reduced form in (a).

Solution to Example 2 Part 4

Since x_3 is the free variable, Let $x_3 = t$ and hence the general solution is

$$\begin{cases} x_1 &= -80 + t \\ x_2 &= 420 - 2t \\ x_3 &= t \end{cases}$$

where t is any integer such that $80 \leq t \leq 210$.

Solution to Example 2 Part 5

If the four-person boat is no longer produced, we should drop the variable x_3 in the linear system i.e. the corresponding augmented matrix should be

$$\left[\begin{array}{cc|c} 0.5 & 1 & 380 \\ 0.6 & 0.9 & 330 \\ 0.2 & 0.3 & 120 \end{array}\right]$$

Using the steps of Gauss-Jordan elimination in (a), we obtain

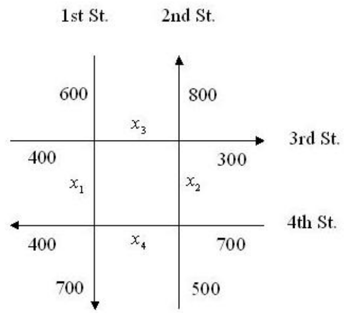
$$\left[\begin{array}{cc|c} 1 & 0 & -80 \\ 0 & 1 & 420 \\ 0 & 0 & 100 \end{array}\right]$$

Part of the reduced form in (a). Therefore, the linear system has no solution.

Example 3 - Traffic Flow

Example

The rush-hour traffic flow for a network of for one-way streets in a city is shown in the figure. The numbers next to each street indicate the number of vehicles per hour that enter and leave the network on that street. The variable x_1, x_2, x_3 and x_4 represent the flow of traffic between the four intersections in the network.



Solution to Example 3 Part 1 (Optional)

(a) For a smooth traffic flow, the number of vehicles entering each intersection should always equal the number leaving. For example, since 1000 vehicles enter the intersection of 1st Street and 3rd Street each hour and $x_1 + x_3$ vehicles leave this intersection, we see that $x_1 + x_3 = 1000$. Find the equations determined by the traffic flow at each of the other three intersections.

(b) Find the solution to the system in (a).

(c) What is the maximum number of vehicles that can travel from 3rd Street to 4th Street on 1st Street? What is the minimum number?

(d) If traffic lights are adjusted so that 700 vehicles per hour travel from 3rd Street to 4th Street on 1st Street, determine the flow around the rest of the network.

Solution to Example 3 Part 2 (Optional)

For each of the four intersections, the number of vehicles entering the intersection equals the number leaving. Therefore, we have

$$\begin{cases} x_1 & & & + & x_4 & = & 1100 \\ x_1 & & + & x_3 & & = & 1000 \\ & x_2 & + & x_3 & & = & 1100 \\ & x_2 & & + & x_4 & = & 1200 \end{cases}$$

Solution to Example 3 Part 3 (Optional)

We use Gauss-Jordan elimination to solve the linear system in (a):

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1100 \\ 1 & 0 & 1 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1100 \\ 0 & 1 & 0 & 1 & 1200 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1100 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 1 & 1 & 0 & 1100 \\ 0 & 1 & 0 & 1 & 1200 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1100 \\ 0 & 1 & 1 & 0 & 1100 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 1 & 0 & 1 & 1200 \end{array}\right] \xrightarrow{-R_2+R_4 \longrightarrow R_4}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1100 \\ 0 & 1 & 1 & 0 & 1100 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & -1 & 1 & 100 \end{array}\right] \xrightarrow[R_3+R_4 \longrightarrow R_4]{-R_3+R_2 \longrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1100 \\ 0 & 1 & 0 & 1 & 1200 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution to Example 3 Part 4 (Optional)

The linear system corresponding to the reduced form is

$$\begin{cases} x_1 \quad\quad\quad + x_4 = 1100 \\ \quad x_2 \quad\quad + x_4 = 1200 \\ \quad\quad x_3 - x_4 = -100 \end{cases}$$

x_4 is the free variable. Hence let $x_4 = t$ and we have the following general solution:

$$\begin{cases} x_1 &= 1100 - t \\ x_2 &= 1200 - t \\ x_3 &= -100 + t \\ x_4 &= t \end{cases}$$

where t is any real number.

Solution to Example 3 Part 5 (Optional)

Similar to the previous example, the solution makes sense only when x_1, x_2, x_3 and x_4 are non-negative integers. Therefore, we have

$$\begin{cases} 1100 - t &\geq 0 \\ 1200 - t &\geq 0 \\ 100 + t &\geq 0 \\ t &\geq 0 \end{cases}$$

and t is an integer. Solving the inequalities, we obtain the all the relevent solutions:

$$\begin{cases} x_1 &= 1100 - t \\ x_2 &= 1200 - t \\ x_3 &= -100 + t \\ x_4 &= t \end{cases}$$

where t is any integer such that $100 \leq t \leq 1100$.

Solution to Example 3 Part 6 (Optional)

(c) By the result from (b), we know that $x_1 = 1100 - t$ and t is any integer such that $100 \leq t \leq 1100$. Therefore, the maximum value of x_1 is 1000 and the minimum value of x_1 is 0.

(d) Set $x_1 = 700$. By the result from (b), $700 = x_1 = 1100 - t$. Hence $t = 400$ and we have

$$\begin{cases} x_1 &= 700 \\ x_2 &= 1200 - 400 = 800 \\ x_3 &= -100 + 400 = 300 \\ x_4 &= 400 \end{cases}$$