MATH 1003 Calculus and Linear Algebra (Lecture 9)

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Addition, Subtraction and Scalar Multiplication

Definition
The addition/subtraction of two matrices of the same size is the matrix with elements that are the addition/subtraction of the corresponding elements of the two given matrices. That is to say, if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, then

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$

Definition
The scalar multiplication of a matrix A by a real number k, denoted by kA, is a matrix formed by multiplying each element of A by k. That is to say, if $A = (a_{ij})_{m\times n}$ then

$$kA = (ka_{ij})_{m\times n}$$

Examples

Evaluate the following expressions:

(a) $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 2 \\ -1 & 6 & -5 \end{bmatrix}$

(b) $-4\begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -14 \\ -3 & 5 \\ -8 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 0 & -2 \\ 1 & -3 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$ does not make sense.

(d) $\begin{cases} x_1 = t - 17 \\ x_2 = -2t + 41 \\ x_3 = t \end{cases} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -17 \\ 41 \\ 0 \end{pmatrix}$

Some Properties

Let A, B and C are matrices of the same size and k be a real number.

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $-A = (-1)A$
- $A + (-1)A = A - A = 0$, where 0 is the zero matrix, a matrix with all its elements being zero.
- $k(A + B) = kA + kB$

Product of a Row Matrix and a Column Matrix

The product of a $1 \times n$ row matrix and an $n \times 1$ column matrix is a 1×1 matrix given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Example

$$\begin{pmatrix} 3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3x_1 + 5x_2 + 4x_3.$$

Matrix Product

- If the matrices $A = (a_{ij})$ is $m \times n$, and $B = (b_{ij})$ is $n \times p$,
- then the matrix product $C = AB = (c_{ij})$ is an $m \times p$ matrix, where

$$c_{ij} = \text{the } i\text{-th row of } A \cdot \text{the } j\text{-th column of } B,$$

that is,

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

- If the number of columns in A does not equal to the number of rows in B, then matrix product AB does not make sense.

Example

Find the value for a_{21} from the product of the following two matrices A and B:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}$$

Matrix Product

Solution

1. Check whether AB is well defined: A is a 2-by-3 matrix and B is a 3-by-3 matrix, well-defined.
2. Find a_{ij} by using the product of the ith-row from the first matrix and the j-th column from the second matrix.

$$\text{2nd row from } A: \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad \text{1st column from } B: \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$a_{21} = 0 \times 2 + 1 \times 0 + 1 \times (-1) = -1.$$

Matrix Product - Examples

Evaluate the following matrix products:

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = ?$

(b) $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = ?$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = ?$

Matrix Product - Examples

Evaluate the following matrix products:

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} Aa + Cb & Ba + Db \\ Ac + Cd & Bc + Dd \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix Product - Examples

(d) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} = ?$

(e) $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} = ?$

(f) $\begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = ?$

(g) $\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} = ?$

Matrix Product - Examples

(d) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}$ not defined!

(f) $\begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = -16$

(g) $\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 15 & 0 \\ 4 & -6 & 0 \\ -4 & 6 & 0 \end{bmatrix}$

Some Properties

Let A and C be two $m \times p$ matrices. Let B and D be two $p \times n$ matrices.

- In general, $AB \neq BA$.
- In general, $AB = 0$ does not imply that $A = 0$ or $B = 0$.
- $A(B + D) = AB + AD$.
- $(A + C)B = AB + CB$.

An Application of Matrix Product

A nutritionist for a cereal company blends two cereals in three different mixes. The amounts of protein, carbohydrate and fat (in grams per ounce) in each cereal are given below.

	Cereal A	Cereal B
Protein	4g/oz	2g/oz
Carbohydrate	20g/oz	16g/oz
Fat	3g/oz	1g/oz

	Mix X	Mix Y	Mix Z
Cereal A	15 oz	10 oz	5 oz
Cereal B	5 oz	10 oz	15 oz

(a) Find the amount of protein in Mix X.

(b) Find the amount of fat in Mix Z.

Solution

We represent two tables by the following matrices:

$$M = \begin{bmatrix} 4 & 2 \\ 20 & 16 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 15 & 10 & 5 \\ 5 & 10 & 15 \end{bmatrix}$$

Then we have

$$MN = \begin{bmatrix} 70 & 60 & 50 \\ 380 & 360 & 340 \\ 50 & 40 & 30 \end{bmatrix}$$

	Mix X	Mix Y	Mix Z
Protein	(a) 70 g	60 g	50 g
Carbonhydrate	380 g	360 g	340 g
Fat	50 g	40 g	(b) 30 g

Summary

Operation		Input		Outcome	
Function	Expression	Input 1	Input 2	size	c_{ij} from
Add / Substraction	C = A ± B	$(a_{ij})_{m\times n}$	$(b_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	a_{ij}, b_{ij}
Scalar multiplication	C = kA	k	$(a_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	k, a_{ij}
Matrix Product	C = AB	$(a_{ij})_{m\times p}$	$(b_{ij})_{p\times n}$	$(c_{ij})_{m\times n}$	i-th row of A, j-th column of B

Take-home Question

Express the following system of linear equations by means of matrix operations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$$