MATH 1003 Calculus and Linear Algebra (Lecture 10)

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Basic Operations Between Numbers

Given two non-zero numbers a and b, one can define

$$a \pm b \text{ (addition/subtraction)}, \qquad ab \text{ (multiplication)}.$$

Similar operations have been defined for two matrices. Now we consider the quotient of two numbers, that is

$$b/a = b \cdot a^{-1},$$

where $a^{-1}a = 1$. a^{-1} is called the inverse of a.
To find the inverse of a matrix, we need to define "1" in the world of matrices first.

Identity Matrix

Definition
The identity matrix of order n is given by the following $n \times n$ matrix:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- We denote the identity matrix by I, or I_n if it is the $n \times n$ identity matrix.

Example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = ?$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = ?$$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example

Theorem

- *$IM = MI = M$ for any square matrix M.*
- *In general, if I is the identity of order n, then*

$$IM = M, \quad NI = N,$$

where M is any $n \times m$ matrix and N is any $p \times n$ matrix.

Inverse of a Matrix

Definition

Let M be a square matrix of order n (i.e., $n \times n$ matrix) and I be the identity matrix of order n. If there exists a matrix N such that

$$NM = MN = I,$$

then N is called the inverse of M.

Remark

- Conventionally, the inverse of M is denoted by M^{-1}.
- M^{-1} is of the same size as M.

Inverse of a Matrix of Order Two

Theorem

When $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $M^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $D = ad - bc$, provided that $D \neq 0$.

Remark

D is called the determinant of M.

Example

Find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

Inverse of a Matrix of Order Two

Solution

By definition, $D = 2 \times 1 - 1 \times 3 = -1 \neq 0$. By the previous theoerem, we have

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

A General Method to Find Matrix Inverse

Theorem

Let M be an $n \times n$ matrix and I be the $n \times n$ identity matrix. If $(M|I)$ can be transformed by row operation(s) into $(I|B)$, then the resulting matrix B is the inverse of M, that is $M^{-1} = B$.

$$(M|I) \xrightarrow{\text{Row operations}} (I|M^{-1}).$$

A General Method to Find Matrix Inverse

Example

Find the inverse of each of the following matrices

(a) $\begin{bmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution for (a) Part 1

$$\left[\begin{array}{ccc|ccc} -5 & -2 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -2 & -2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3+5R_1 \to R_3]{R_2+(-2)R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & -2 & 3 & 1 & 0 & 5 \end{array}\right] \xrightarrow{R_3+2R_2 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{-R_3 \to R_3}$$

Solution for (a) Part 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & -2 & -1 \end{array}\right] \xrightarrow[R_1+(-1)R_3 \to R_1]{R_2+2R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & -2 & -3 & -4 \\ 0 & 0 & 1 & -1 & -2 & -1 \end{array}\right]$$

Therefore, $\begin{bmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution for (b)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3+(-2)R_1 \to R_3]{R_2+(-1)R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{array}\right]$$

There is a row of zeroes on the left matrix and hence it is impossible to transform it into an identity matrix by row operations. No inverse exists.

Matrix Operation in Microsoft Excel (Optional)

1. Select a set of cells with the same configuration to the expected outcome (e.g. 3×3)
2. Type in the command
   - Inverse "=MINVERSE(A1:C3)";
   - Multiplication "=MMULT(A1:C3,A5:C7)"
3. Press "Ctrl"+"Shift"+"Enter"