MATH 1003 Calculus and Linear Algebra (Lecture 11)

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Properties for Matrix Operation

Addition (A, B and C of same size)

- Associative: $(A+B)+C = A+(B+C)$
- Commutative: $A+B = B+A$

Multiplication (A of size $m \times p$; B of size $p \times q$; C of size $q \times n$)

- Associative: $(AB)C = A(BC)$
- Not Commutative: $AB \neq BA$, even when both AB and BA are meaningful.

Combined Properties

- Left distributive: $A(B+C) = AB + AC$
- Right distributive: $(B+C)A = BA + CA$

Similar to the operations of numbers except that multiplication is not commutative

Matrix Equations

Number equation

If a, x and b are numbers and given a invertible,

$$ax = b \quad \to \quad x = a^{-1}b.$$

Matrix equation

For matrices, given A of size $n \times n$, an unknown matrix X of size $n \times p$ and B of size $n \times p$,

$AX = B$	To find X
$A^{-1}(AX) = A^{-1}B$	Left multiply A^{-1} on both sides
$(A^{-1}A)X = A^{-1}B$	Associate A^{-1} with A
$IX = A^{-1}B$	Properties of inverse
$X = A^{-1}B$	Properties of identity matrix

Matrix Equations - an Example

Solve the following system

$$\begin{cases} x_1 - x_2 + x_3 &= 1 \\ \quad 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 &= 1 \end{cases}$$

from the matrix-equation form:

$$AX = B.$$

Matrix Equations - Solution Part 1

By comparison, we have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find inverse of A:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[(-5)R_2+R_3\to R_3]{R_2+R_1\to R_1}$$

Matrix Equations - Solution Part 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -2 & -\frac{5}{2} & 1 \end{array}\right] \xrightarrow[-R3+R_2\to R_2]{-R3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & \frac{1}{2} & -2 & -\frac{5}{2} & 1 \end{array}\right]$$

$$\xrightarrow{2R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right]$$

Therefore, the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -7 \end{bmatrix}$$

Generalisation of the Above Example

Example

Use matrix inverse to solve the system

$$\begin{cases} x_1 - x_2 + x_3 &= k_1 \\ 2x_2 - x_3 &= k_2 \\ 2x_1 + 3x_2 &= k_3 \end{cases}$$

Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 3k_1 + 3k_2 - k_3 \\ -2k_1 - 2k_2 + k_3 \\ -4k_1 - 5k_2 + 2k_3 \end{bmatrix}$$

An Application

Example

An investment advisor currently has two types of investment available for clients: a conservative investment A that pays 10% per year and an investment B of higher risk that pays 20% per year. Clients may divide their investments between the two to achieve any total return desired between 10% and 20%. However, the higher the desired return, the higher the risk. How should each client listed in the table invest to achieve the indicated return?

	Client 1	Client 2	Client 3
Total investment	\$20,000	\$50,000	\$10,000
Annual Return Desired	\$2,400	\$7,500	1,300

Solution Part 1

Let x_1 and x_2 be the amount of investment A and B that a client has respectively.

According to the given table, all three clients have different total investment and annual return desired. We denote the total investment and annual return desired by k_1 and k_2 respectively. Then we have

$$\begin{cases} x_1 + x_2 &= k_1 \\ 0.1x_1 + 0.2x_2 &= k_2 \end{cases}$$

We rewrite it as the following matrix equation:

$$\begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Solution Part 2

We need to find the inverse:

$$\begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix}^{-1} = \frac{1}{0.1} \begin{bmatrix} 0.2 & -1 \\ -0.1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix}$$

Hence, we can use the inverse to solve the system:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2k_1 - 10k_2 \\ -k_1 + 10k_2 \end{bmatrix}$$

Therefore, we have

- Client 1 ($k_1 = 20000$, $k_2 = 2400$): $x_1 = 16000$, $x_2 = 4000$.
- Client 2 ($k_1 = 50000$, $k_2 = 7500$): $x_1 = 25000$, $x_2 = 25000$.
- Client 3 ($k_1 = 10000$, $k_2 = 1300$): $x_1 = 7000$, $x_2 = 3000$.

Summary

We have two major methods to solve a system of linear equations. They are

1. Gauss-Jordan Elimination
   - Capable of dealing with a system whose number of equations does not equal to that of variables.
   - If there are infinite number of solutions, it is easier to write the general form for the solutions by G-J elimination .
2. Inverse Method
   - A number of system of linear equations sharing a same matrix.
   - The matrix has to be square (equal number of equations and variables).