MATH 1003 Calculus and Linear Algebra (Lecture 12)

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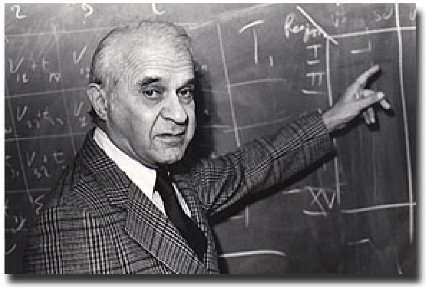
Leontief Input-Output Analysis

Wassily Leontief was a Russian-American economist who pioneered the analysis of the so-called input-output model - a model that describes how the input and output of different industries affect each other. The theory involves the extensive use of matrices.



Two-Industry Model

The following is a description of a two-industry model:

- We have two industries, electric company E and water company W. Output for both companies is measured in dollars.
- The electric company uses both electricity and water (input) in the production of electricity (output), and the water company uses both electricity and water (input) in the production of water (output).
- Suppose that the production of each dollar's worth of electricity requires \$0.3 worth of electricity and \$0.1 worth of water, and the production of each dollar's worth of water requires \$0.2 worth of electricity and \$0.4 worth of water.
- The final demand from the outside sector of the economy is \$12 million for electricity and \$8 million for water.

Question: How much electricity and water should be produced to meet this final demand?

Solution to Two-Industry Model, Part 1

Let x_1 and x_2 be the total output of E and W respectively.

An observation: x_1 and x_2 should be greater than the final demand of E and W respectively because the input of E and W, which we call the internal demand, is needed in order to produce the final demand. Therefore, for both E and W,

$$\text{Total output} = \text{Internal demand} + \text{Final demand}$$

How can we find the internal demand of E and W?

Solution to Two-Industry Model, Part 2

We are given the following information:

- \$0.3 of E and \$0.1 of W generates \$1 of E, and
- \$0.2 of E and \$0.4 of W generates \$1 of W.

Obviously, we can deduce that

- \$$0.3x_1$ of E and \$$0.1x_1$ of W generates \$$x_1$ of E, and
- \$$0.2x_2$ of E and \$$0.4x_2$ of W generates \$$x_2$ of W.

Hence,

- the internal demand of E is \$$(0.3x_1 + 0.2x_2)$, and
- the internal demand of W is \$$(0.1x_1 + 0.4x_2)$.

Solution to Two-Industry Model, Part 3

The system for x_1 and x_2 is as follows:

$$\begin{cases} x_1 &= 0.3x_1 + 0.2x_2 + 12 \\ x_2 &= 0.1x_1 + 0.4x_2 + 8 \end{cases}$$

Now, we rewrite the system as a matrix equation. Let

$$M = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad D = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

Then the system becomes

$$X = MX + D$$

Solution to Two-Industry Model, Part 4

Now we need to solve the matrix equation for X:

$$X - MX = D$$

$$\Rightarrow IX - MX = D$$

$$\Rightarrow (I - M)X = D$$

$$\Rightarrow X = (I - M)^{-1}D$$

That is to say, we need to compute the inverse of $I - M$ in order to solve the system.

Solution to Two-Industry Model, Part 5

$$I - M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}$$

$$(I - M)^{-1} = \frac{1}{0.4} \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}$$

Therefore, $X = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix}$

Hence,

- the total output of E is \$22 million, and
- the total output of W is \$17 million.

General Two-Industry Model

Suppose that we have two industries, A and B.

- The production of each dollar's worth of A requires $\$m_{11}$ worth of A and $\$m_{21}$ worth of B.
- The production of each dollar's worth of B requires $\$m_{12}$ worth of A and $\$m_{22}$ worth of B.
- The final demand from the outside sector of the economy is $\$d_1$ million for A and $\$d_2$ million for B.

Let x_1 and x_2 be the total output from A and B respectively, then we have

$$X = MX + D, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

If $I - M$ is invertible, then the solution for X is given by

$$X = (I - M)^{-1} D.$$

We call D the final demand matrix, X the output matrix and M the technology matrix.

Three-Industry Model, an Example

An economy is based on three sectors, agriculture (A), energy (E) and manufacturing (M).

- Production of a dollar's worth of agriculture requires an input of \$0.2 from the agriculture sector and \$0.4 from the energy sector.
- Production of a dollar's worth of energy requires an input of \$0.2 from the energy sector and \$0.4 from the manufacturing sector.
- Production of a dollar's worth of manufacturing requires an input of \$0.1 from the agriculture sector, \$0.1 from the energy sector, and \$0.3 from the manufacturing sector.

Find the output from each sector that is needed to satisfy a final demand of \$20 billion for agriculture, \$10 billion for energy, and \$30 billion for manufacturing.

Solution to Three-Industry Model, Part 1

Let x_1, x_2 and x_3 be the total output of A, E and M respectively.

Then the technology matrix $M = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.4 & 0.2 & 0.1 \\ 0 & 0.4 & 0.3 \end{bmatrix}$

Therefore, $I - M = \begin{bmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{bmatrix}$

Then by computation, $(I - M)^{-1} = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix}$

Solution to Three-Industry Model, Part 2

We have

$$X = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix} = \begin{bmatrix} 33 \\ 37 \\ 64 \end{bmatrix}$$

Hence,

- the total output of A is \$33 million,
- the total output of E is \$37 million, and
- the total output of M is \$64 million.

n-Industry Model

For n inter-dependent industries, technology matrix M is

To produce		A	B	C	⋯
Need	A	m_{11}	m_{12}	m_{13}	⋯
	B	m_{21}	m_{22}	m_{23}	⋯
	C	m_{31}	m_{32}	m_{33}	⋯
	⋯	⋯	⋯	⋯	⋯

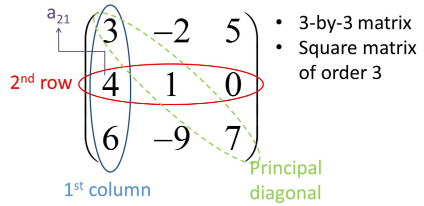
- The final demand matrix D and the output matrix X are $n \times 1$ matrices.
- The matrix equation for the model is

$$X = MX + D$$

- Therefore, $X = (I - M)^{-1}D$ if $I - M$ is invertible.

Summary of Knowledge for Matrices

- System of linear equations in 2 variables (L.5)
  - Algebraic form v.s. Geometric form
  - supply and demand curve
- Basic ideas about Matrices (L.6)
  - To know a matrix



  - Row operation

$$R_i \leftrightarrow R_j, \quad kR_i \to R_i, \quad R_i + kR_j \to R_i$$

Summary of Knowledge for Matrices

- Gauss-Jordan Elimination
  - Solver to system of linear equations (method 1, L.7)
  - Using row operation to get the reduced form (L.7)
  - System with infinite number of solutions which belong to several families (L.7)
  - Application: variables and restrictions, positive numbers may be required for real world problem (L.8)
- Matrix Operation (L.9)

Operation		Input		Outcome	
Function	Expression	Input 1	Input 2	size	c_{ij} from
Add / Substraction	$C = A \pm B$	$(a_{ij})_{m\times n}$	$(b_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	a_{ij}, b_{ij}
Scalar multiplication	$C = kA$	k	$(a_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	k, a_{ij}
Matrix Product	$C = AB$	$(a_{ij})_{m\times p}$	$(b_{ij})_{p\times n}$	$(c_{ij})_{m\times n}$	i-th row of A, j-th column of B

Summary of Knowledge for Matrices

- Inverse Matrix (L.10)
  - Identity matrix I

$$IM = MI = M$$

  - Given a square matrix M, its inverse matrix M^{-1} satisfies

$$MM^{-1} = M^{-1}M = I$$

  - Find M^{-1} (order 2, order 3), not every square matrix is invertible
- Matrix Equation
  - Solver to system of linear equations (method 2), unknown right hand side (L.11)
  - Leontief input-output analysis (L.12)