MATH 1003 Calculus and Linear Algebra (Lecture 13)

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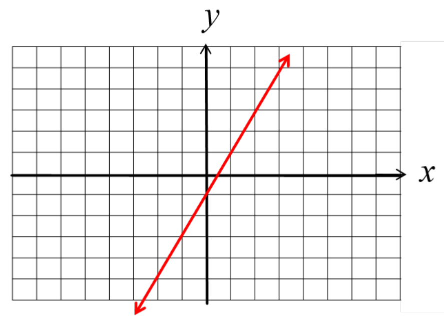
Graph of a function

The graph of the function $y = f(x)$ is the graph of the set of all ordered pairs $(x, f(x))$.
Example: Graph the function $f(x) = 2x - 1$

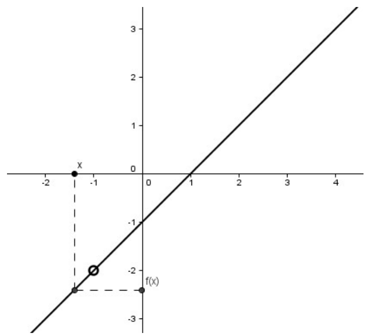


x	-2	-1	0	1	2
$f(x)$	-5	-3	-1	1	3

Limits

Example

Graph the function $f(x) = \dfrac{x^2 - 1}{x + 1}$. Then find out how $f(x)$ behaves as x tends to, but not equal to -1.



Limits

Remarks

- As x gets close to -1, $f(x)$ gets close to -2.
- We say that the limit of $f(x)$ is -2 as x tends to -1.
- Notice that $f(x)$ is undefined when $x = -1$.
- The limit of $f(x)$ as x tends to -1 is well-defined even when $f(x)$ is undefined at -1. In fact, the value of f at -1 does not affect the limit.

Definition of the Limit of a Function

Definition

We write

$$\lim_{x\to c} f(x) = L \quad \text{or} \quad f(x)\to L \quad \text{as} \quad x\to c$$

if $f(x)$ is close to the single real number L when x tends to, but not equal to c.

Remarks

- Limit does not always exist i.e. it may happen that $f(x)$ does not get close to any real number as x tends to a certain c.
- Limit is unique if it exists.

The limit of a function $f(x)$ at c is determined by the evaluation of $f(x)$ in the "neighbourhood" of c and may have nothing to do with $f(c)$.

One-sided Limits

Example

Let $h(x) = \dfrac{|x|}{x}$. Find $\lim\limits_{x\to 0} h(x)$.

Note that $h(0)$ is not defined, and

$$|x| = \begin{cases} x & : \quad x \ge 0 \\ -x & : \quad x < 0 \end{cases}$$

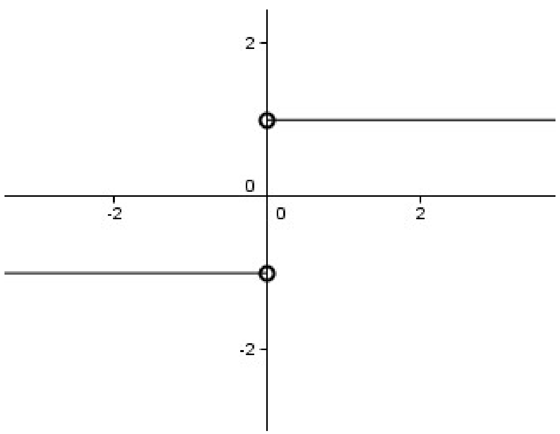
Hence

$$h(x) = \begin{cases} \frac{x}{x} = 1 & : \quad x > 0 \\ \text{not defined} & : \quad x = 0 \\ \frac{-x}{x} = -1 & : \quad x < 0. \end{cases}$$

One-sided Limits

Example

Let $h(x) = \dfrac{|x|}{x}$. Find $\lim\limits_{x\to 0} h(x)$. (The graph of $y = h(x)$ is as follows.)



One-sided Limits

Remarks

- As x gets close to 0 from the right hand side, $h(x)$ gets close to 1.
- As x gets close to 0 from the left hand side, $h(x)$ gets close to -1.
- If x gets close to 0 without restricting the direction of approach, $h(x)$ cannot be close to a single value.

We saw that the values of the function $h(x)$ approached two different numbers, depending on the direction of approach, and it was natural to refer to these value as "the limit from the left" and "the limit from the right".

Left-hand and Right-hand Limits

Example

Find $\lim_{x\to 0^+} h(x)$, $\lim_{x\to 0^-} h(x)$ and $\lim_{x\to 0} h(x)$, where $h(x) = \frac{|x|}{x}$.

Solution

- $\lim_{x\to 0^+} h(x) = 1$.
- $\lim_{x\to 0^-} h(x) = -1$.
- $\lim_{x\to 0} h(x)$ does not exist.

Theorem

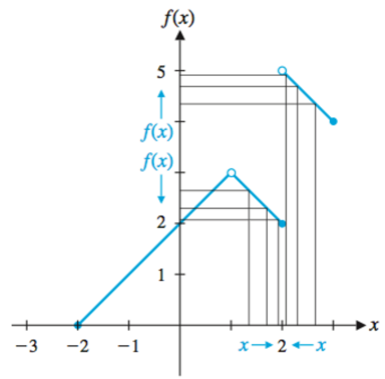
A limit exists means that the left-hand limit and the right-hand limit must be equal. That is,

$$\lim_{x\to c} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L$$

One-sided Limits

Example

For the graph of a function below, find
$\lim_{x\to 2^-} f(x), \lim_{x\to 2^+} f(x), \lim_{x\to 2} f(x), f(2)$ and
$\lim_{x\to 1^-} f(x), \lim_{x\to 1^+} f(x), \lim_{x\to 1} f(x), f(1)$.



(Answer: 2, 5, not exist, 2; 3, 3, 3, not exist.)

Finding limits without graph

Example

Find $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(x+1)^2 - 1}{x}$.

Solution

The idea is to evaluate $f(x)$ for various values of x which are close to 0 e.g. $x = 0.01, 0.001, -0.01, -0.001$ and so on to guess the limit. Consider the following table:

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	1.99	1.999	1.9999	2.0001	2.001	2.01

Notice that $f(x)$ tends to 2 as x tends to 0. Therefore,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(x+1)^2 - 1}{x} = 2$$

Finding limits without graph

Remarks

- In the previous example, $f(x)$ can in fact be simplified in order to make the calculation easier:

$$f(x) = \frac{(x+1)^2 - 1}{x} = \frac{x^2 + 2x}{x} = x + 2$$

whenever $x \neq 0$. Therefore, it is clear that $\lim_{x\to 0} f(x) = 2$.

- When evaluating one-sided limit, say $\lim_{x\to 0^+} f(x)$, the values of x we use must be on one side i.e. $x = 0.01, 0.001$ and so on. (Note: for $\lim_{x\to 0^-} f(x)$, we use $x = -0.01, -0.001$ and so on)

Properties of Limits

Theorem

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (both limits exist), then

1. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kL$ *if k is a constant,*
2. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm M,$
3. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = LM,$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ *if $M \neq 0$*
5. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ *if $L > 0$ for n even.*

Theorem

1. $\lim_{x\to c} k = k$ *for any constant k,*
2. $\lim_{x\to c} x = c,$
3. $\lim_{x\to c} f(x) = f(x)$ *for any polynomial function f.*
4. $\lim_{x\to c} r(x) = r(c)$ *for any rational function r with a nonzero denominator at $x = c$.*

Example

Example

1. $\lim_{x\to 3}(x^2 - 4x) = 3^2 - 4 * 3 = -3.$
2. $\lim_{x\to 2}(x^3 - 5x - 1) = 2^3 - 5 * 2 - 1 = -3.$
3. $\lim_{x\to -1} \sqrt{2x^2 + 3} = \sqrt{2 * (-1)^2 + 3} = \sqrt{5}.$
4. $\lim_{x\to 4} \frac{2x}{3x+1} = \frac{2*4}{3*4+1} = \frac{8}{13}.$

Example

Example

Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$, $f(x)$.

(Answer: $5, 1$, does not exist, does not exist.)

Limits

Theorem

If $\lim_{x\to c} f(x) = L, L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} \quad \textit{does not exist.}$$

Definition

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ is said to be indeterminate, or more precisely, a $0/0$ indeterminate form. (The limit may or may not exist and can be determined by other methods.)

Limits

Example

1. $\lim_{x\to1}\frac{x-1}{x^2+1}=\frac{1-1}{1^2+1}=\frac{0}{2}=0$.
2. $\lim_{x\to1}\frac{x^2+1}{x-1}=\frac{1^2+1}{1-1}=\frac{1}{0}$, the limit does not exist.
3. $\lim_{x\to1}\frac{x-1}{x^2-1}=\frac{1-1}{1^2-1}=\frac{0}{0}$, indeterminate. Actually $\lim_{x\to1}\frac{x-1}{x^2-1}=\lim_{x\to1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to1}\frac{1}{x+1}=\frac{1}{2}$.
4. $\lim_{x\to1}\frac{x-1}{(x-1)^2}=\frac{1-1}{(1-1)^2}=\frac{0}{0}$, indeterminate. Actually $\lim_{x\to1}\frac{x-1}{(x-1)^2}=\lim_{x\to1}\frac{1}{x-1}=\lim_{x\to1}\frac{1}{0}$, the limit does not exist.

Take-home exercises

Example

Find the limit for

$$\lim_{h\to0}\frac{f(3+h)-f(3)}{h}$$

for the following functions

1. $f(x)=4x-5$,
2. $f(x)=x^2+3x$,
3. $f(x)=\frac{1}{x}$.

(Answer: $4, 12, -\frac{1}{9}$)