MATH 1003 Calculus and Linear Algebra (Lecture 13.5)

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Continuity

Definition

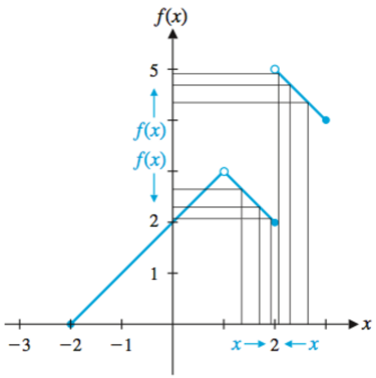
A function $f : (a, b) \to \mathbb{R}$ is continuous at a point $c \in (a, b)$ if

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ exists,
3. $\lim_{x\to c} f(x) = f(c)$.

$f(x)$ is called continuous over (a, b) if f is continuous at every point $c \in (a, b)$.

Example 0

Example



The function $f(x)$ is discontinuous at $x = 1, 2$ but is continuous at all other points: $f(x)$ is not defined at $x = 1$, and $\lim_{x\to 2} f(x)$ does not exist.

Example 1

Example

Is the function $f(x)$ continuous at the point?

1. $f(x) = x - 1$ at $x = 2$.
2. $f(x) = \begin{cases} x^2 - 1 & \text{if } x > 2 \\ 3 & \text{if } x = 2 \\ x + 1 & \text{if } x < 2 \end{cases}$, at $x = 2$.

Solution 1. Since $\lim_{x\to 2}(x - 1) = 2 - 1 = 1$ and $f(2) = 2 - 1 = 1$, they are equal, so f is continuous at $x = 2$.
2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x + 1) = 2 + 1 = 3$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^2 - 1) = 2^2 - 1 = 3$, $f(2) = 3$, they are equal, so f is continuous at $x = 2$.
Remark For question 2, if $f(2)$ is changed to 5, then f is not continuous at $x = 2$.

Continuity properties

Theorem

1. *A polynomial function is continuous for all x ($f(x) = 3x + 5 + 8x^{10}$ is continuous for all x).*
2. *A rational function is continuous for all x except those values that make a denominator 0. ($f(x) = \frac{x^2+5}{x-1}$ is continuous for all $x \neq 1$)*
3. *If n is an odd positive integer, then $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous. ($\sqrt[3]{x^4}$ is continuous for all x)*
4. *If n is an even integer, then $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative. ($\sqrt[4]{x}$ is continuous on $[0, \infty)$)*

Continuous functions

Theorem

If $f(x)$ is continuous on (a, b) and $f(x) \neq 0$ for all x in (a, b), then either $f(x) > 0$ for all x in (a, b) or $f(x) < 0$ for all x in (a, b).

In other words, if $f(x_1) < 0$ and $f(x_2) > 0$ for a continuous function f, then there exists x_0 such that $f(x_0) = 0$.

Solving inequality and sign charts

Example

Find the range of x such that:

1. $\frac{x+1}{x-2} > 0$ (Answer $(-\infty, -1) \cup (2, \infty)$)
2. $\frac{x^2-1}{x-3} < 0$ (Answer $(-\infty, -1) \cup (1, 3)$)
3. $\frac{x^2+1}{x-3} > 0$ (Answer $(3, \infty)$)

Average Rate of Change

Let $y = f(x)$. Roughly speaking, the "rate of change" of y is

$$\frac{\text{Change in } y}{\text{Change in } x}$$

More rigorously, we have the following definition:

Definition

For $y = f(x)$, the average rate of change from $\mathbf{x} = \mathbf{a}$ to $\mathbf{x} = \mathbf{b}$ is

$$\frac{f(b) - f(a)}{b - a}$$

Average Rate of Change

Example

A small ball dropped from a tower will fall a distance of y feet in x seconds, as given by the formula

$$y = 16x^2.$$

(a) Find the average velocity from $x = 2$ seconds to $x = 3$ seconds.

(b) Find the average velocity from $x = 2$ seconds to $x = 2 + h$ seconds, $h \neq 0$.

(c) Find the expression from part (2) as $h \to 0$, if it exists.

Average Rate of Change

Solution

(a) The average velocity is $\dfrac{16 \times 3^2 - 16 \times 2^2}{3-2} = 80$ feet per second.

(b) The average velocity is $\dfrac{16(2+h)^2 - 16 \times 2^2}{2+h-2} = \dfrac{16(h^2+4h)}{h}$ feet per second.

(c) $\lim\limits_{h \to 0} \dfrac{16(h^2+4h)}{h} = \lim\limits_{h \to 0} 16(h+4) = 64$ feet per second.

Instantaneous Rate of Change

In the previous example, we consider the average rate of change of distance when the change of x is from 2 to $2 + h$. And then we let h tends to 0. The limit can be regarded as the "instantaneous rate of change of at 2". In general, we have the following definition:

Definition

For $y = f(x)$, the instantaneous rate of change at $\mathbf{x = a}$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

i.e. it is the limit of the difference quotient of f at $x = a$.

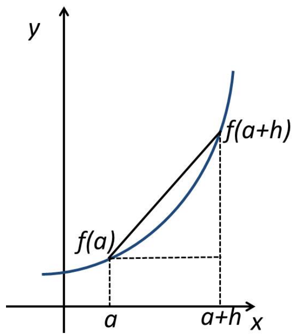
Example

The instantaneous rate of change at $x = 2$ in the previous example is 64.

Slope of a Secant Line

A line through two point on the graph of $y = f(x)$ is called a secant line. If $(a, f(a)$ and $(a+h, f(a+h))$ are two points on the graph of $y = f(x)$, then the slope of secant line from $x = a$ to $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$



Thus, the slope of secant line can be interpreted as the average rate of change of y from $x = a$ to $x = a + h$.

Slope of a Secant Line

Example

Given $y = f(x) = 0.5x^2$,

(a) Find the slope of secant line for $a = 1$, and $h = 2$.

(b) Find the slope of secant line for $a = 1$ and h for any nonzero number.

(c) Find the limit of expression in (b) as $h \to 0$.

Slope of a Secant Line

Solution

(a) The slope of secant line is

$$\frac{f(3) - f(1)}{2} = 2$$

(b) The slope of secant line is

$$\frac{f(1+h) - f(1)}{h} = \frac{0.5(1+h)^2 - 0.5}{h} = \frac{h + 0.5h^2}{h} = 1 + 0.5h$$

(c) As $h \to 0$, we have

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} (1 + 0.5h) = 1$$

Slope of a Tangent

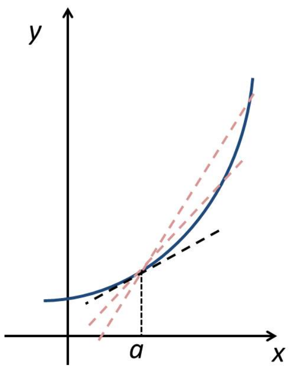
From the graph, we observe that the slope of the secant line tends to the slope of the tangent as h tends to 0. Therefore, we have the following definition:

Definition

Given $y = f(x)$, the slope of the tangent line of $\mathbf{f(x)}$ at the point $\mathbf{x = a}$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.



The Derivative

Definition

For $y = f(x)$, we define the derivative of $\mathbf{f}$ at $\mathbf{x}$, denoted by $\mathbf{f'(x)}$, $\mathbf{\frac{dy}{dx}}$ or $\mathbf{\frac{df}{dx}}$, to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exist. If $f'(x)$ exists for each x in the interval $a < x < b$, then f is said to be differentiable over $a < x < b$.

Summary

There are three different interpretations of the derivative of $f(x)$:

- Limit of the difference quotient: $f'(x)$ is the limit of the different quotient of f at x.
- Slope of the tangent line: $f'(x)$ is the slope of the line tangent to the graph of f at the point $(x, f(x))$.
- Instantaneous rate of change: $f'(x)$ is the instantaneous rate of change of $y = f(x)$ with respect to x.

Exercise

Example

Find $f'(1)$ for each of the following functions:

(a) $f(x) = 2x - x^2$

(b) $f(x) = x^3$

(c) $f(x) = \dfrac{1}{x}$

(d) $f(x) = \sqrt{x}$

Answers: a) 0 b) 3 c) -1 d) 1/2