MATH 1003 Calculus and Linear Algebra (Lecture 15)

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Re-visit to Compound Interest Rate

Example

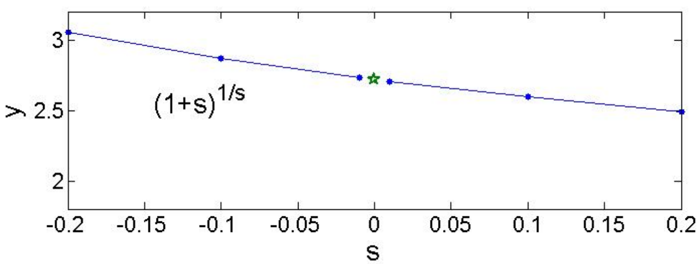
If \$100 is invested at 6%, what amount will be in the account after 2 years with various compounding frequencies?

Compounding Frequency	n	$A = 100\left(1 + \frac{0.06}{n}\right)^{2n}$
Annually	1	112.3600
Semiannually	2	112.5509
Quarterly	4	112.6493
Monthly	12	112.7160
Weekly	52	112.7419
Daily	360	112.7486
Hourly	8640	112.7496

Re-visit to Compound Interest Rate

Calculate $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ or equivalently calculate $\lim_{s\to 0}(1+s)^{\frac{1}{s}}$

s	-0.2	-0.1	-0.01	0.01	0.1	0.2
$(1+s)^{\frac{1}{s}}$	3.0518	2.8680	2.7320	2.7048	2.5937	2.4883



The Constant e

Definition

The number e is defined by

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{s\to 0}(1+s)^{\frac{1}{s}} \approx 2.7182818284590.....$$

If P is invested at an annual rate r, compounded n times per year, then after t years, the amount is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Now we let n go to infinity, we find that ($s = \frac{n}{r}$, so $n = rs$)

$$\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = P\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{(n/r)rt} = P[\lim_{s\to 0}(1+s)^{1/s}]^{rt} = Pe^{rt}.$$

The above formula is called the continuous compound interest formula.

The Constant e

Example

How long will it take to double an investment if it is invested at 5% compounded continuously?

Solution

Let P be the principal. Then we have

$$2P = Pe^{0.05t}$$

$$\Rightarrow t = \frac{\ln 2}{0.05} = 13.86 \text{ years.}$$

Exponential functions

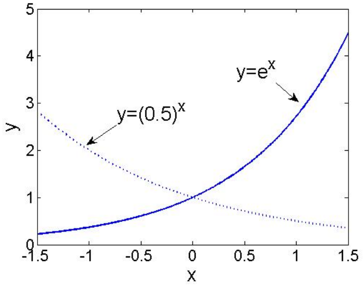
Definition

$f(x) = e^x$ is called the exponential function with base e. More generally, $f(x) = a^x$, where $a > 0$, is called the exponential function with base a. Exponential functions are defined for any real number x.

Remarks

- When $a > 1$, $f(x)$ increases as x increases.
- When $a = 1$, $f(x) = 1$.
- When $0 < a < 1$, $f(x)$ decreases as x increases.

Outlook



Derivative of e^x

Theorem

Let $y = f(x) = e^x$, *then* $\frac{dy}{dx} = f'(x) = e^x$.

Observation

If $f(x) = e^x$, then $f'(x) = f(x)$.

Example

Find $f'(x)$ for each of the following functions:

(a) $f(x) = x^2 - 2e^x$

(b) $f(x) = 3x^e - 6e^x + e^4$

Solutions

(a) $f'(x) = 2x - 2e^x$

(b) $f'(x) = 3ex^{e-1} - 6e^x$

Logarithmic Function

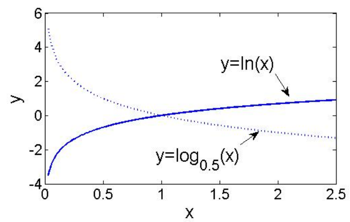
Definition

$g(x) = \ln x$ is called the (natural) logarithmic function with base e. It is only defined for $x > 0$. More generally, $g(x) = \log_a x$, where $a > 0$, is called the logarithmic function with base a.

Remarks

- When $a > 1$, $g(x)$ increases as x increases.
- When $a = 1$, $g(x)$ is not a well-defined function.
- When $0 < a < 1$, $g(x)$ decreases as x increases.

Outlook



Exponential and Logarithmic Functions

Remarks

- Logarithmic functions are the inverse of the corresponding exponential functions:
  $y = e^x \iff x = \ln y$
  $y = a^x \iff x = \log_a y.$
- $e^0 = 1, \ln 1 = 0.$
- $a^0 = 1, \log_a 1 = 0$, for any $a > 0$.
- $\ln a + \ln b = \ln(ab), \quad e^{a+b} = e^a \cdot e^b.$
- $\ln(a^r) = r \ln a, \quad e^{ab} = (e^a)^b.$
- $\ln \frac{1}{a} = \ln a^{-1} = -\ln a.$
- $3 > e \approx 2.718... > 2.$

Derivative of $\ln x$

Theorem

Let $y = g(x) = \ln x$*, then* $\dfrac{dy}{dx} = g'(x) = \dfrac{1}{x}$.

Example

Find $f'(x)$ for each of the following functions:

(a) $f(x) = 5\ln x + 4x^3$

(b) $f(x) = \ln 2 - 4\ln(x^2) + 3$

Solutions

(a) $f'(x) = \dfrac{5}{x} + 12x^2.$

(b) $f'(x) = (-8\ln(x))' = -\dfrac{8}{x}.$

Derivatives of Other Exponential and Logarithmic Functions

To differentiate a general exponential or logarithmic function, we need the following theorem:

Theorem

Let $f(x) = a^x$ *and* $g(x) = \log_a x$*. Then*

$$f'(x) = a^x \ln a \quad \text{and} \quad g'(x) = \frac{1}{x \ln a}.$$

Example

Find $f'(x)$ for each of the following functions:

(a) $f(x) = 3\log_4 x - 5x^3$

(b) $f(x) = 4 \cdot 3^x - 2\log_4 3$

An Application

Solutions

(a) $f'(x) = \dfrac{3}{x \ln 4} - 15x^2.$

(b) $f'(x) = 4 \cdot 3^x \ln 3.$

Example

An Internet store sells blankets made of Iceland wool. If the store sells x blankets at a price of p per blanket, then the price-demand equation is $p(x) = 320(0.998)^x$. Find the rate of change of price with respect to demand when the demand is 800 blankets.

Solutions

$$p'(x) = 320(0.998)^x \ln(0.998)$$

$$\Rightarrow p'(800) = -0.129$$

Summary

Theorem

- $f(x) = e^x, \quad f'(x) = e^x.$
- $f(x) = \ln x, \quad f'(x) = \frac{1}{x}.$
- $f(x) = a^x, \quad f'(x) = a^x \ln a.$
- $f(x) = \log_a x, \quad f'(x) = \frac{1}{x \ln a}.$