MATH 1003 Calculus and Linear Algebra (Lecture 16)

Maosheng Xiong
Department of Mathematics, HKUST

Product Rule

Theorem

(Product Rule) If

$$y = f(x) = u(x)v(x),$$

then

$$f'(x) = u(x)v'(x) + u'(x)v(x).$$

Remark

- In general, $(u(x)v(x))' \neq u'(x)v'(x)$.
- The product rule can be generalized as follows:

$$\begin{aligned}\{u(x)\underline{v(x)w(x)}\}' &= u'(x)\underline{v(x)w(x)} + u(x)\left\{\underline{v(x)w(x)}\right\}' \\ &= u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x).\end{aligned}$$

Product Rule

Example

Find the derivative for each of the following functions:

(a) $f(x) = 2x^2e^x$

(b) $f(x) = (\sqrt{x} + 1)(\ln x - 1)$

Solutions

(a) $f'(x) = (2x^2)'e^x + 2x^2(e^x)' = 4xe^x + 2x^2e^x$.

(b) $f'(x) = (\sqrt{x} + 1)'(\ln x - 1) + (\sqrt{x} + 1)(\ln x - 1)'$. Hence we have

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(\ln x - 1) + (\sqrt{x} + 1)\frac{1}{x}.$$

Quotient Rule

Theorem

(Quotient Rule) If

$$y = f(x) = \frac{u(x)}{v(x)},$$

then

$$\frac{dy}{dx} = f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}.$$

Remark

- In general, $\left(\frac{u(x)}{v(x)}\right)' \neq \frac{u'(x)}{v'(x)}$.
- $u(x)$ and $v(x)$ in the above formula cannot be interchanged.

Quotient Rule

Example

Find the derivative for each of the following functions:

(a) $f(x) = \frac{x}{\sqrt{x}+1}$

(b) $f(x) = \frac{e^x}{2x+1}$

Solutions

(a) $f'(x) = \dfrac{x'(\sqrt{x}+1) - x(\sqrt{x}+1)'}{(\sqrt{x}+1)^2} = \dfrac{\sqrt{x}+1-\frac{1}{2}\sqrt{x}}{(\sqrt{x}+1)^2} = \dfrac{\sqrt{x}+2}{2(\sqrt{x}+1)^2}$.

(b) $f'(x) = \dfrac{(e^x)'(2x+1) - e^x(2x+1)'}{(2x+1)^2} = \dfrac{e^x(2x+1) - 2e^x}{(2x+1)^2} = \dfrac{e^x(2x-1)}{(2x+1)^2}$.

More practices

Example

Find the derivative of each of the following functions:

(a) $f(x) = \dfrac{x^2}{x^2-1}$

(b) $g(t) = \dfrac{t^2-t}{2^t}$

(c) $y = \dfrac{te^t}{\ln t}$

Answers to practices

(a)
$$f'(x) = \frac{(x^2)'(x^2-1) - (x^2-1)'x^2}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2};$$

(b)
$$g'(t) = \frac{(t^2-t)'2^t - (t^2-t)(2^t)'}{(2^t)^2} = \frac{2t-1-(t^2-t)\ln 2}{2^t};$$

(c)
$$\frac{dy}{dt} = \frac{(te^t)'\ln t - (\ln t)'(te^t)}{(\ln t)^2} = \frac{e^t(t+1)\ln t - e^t}{(\ln t)^2}.$$