MATH 1003 Calculus and Linear Algebra (Lecture 17)

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Composite Functions

Definition

If $y = g(v)$ and $v = u(x)$, then $y = f(x)$ is a composite function if

$$y = f(x) = g(u(x))$$

Example

Find $y = f(x) = g(u(x))$ if

(a) $g(v) = v^5, \;\; u(x) = 2x + 1$

(b) $g(v) = 2v + 1, \;\; u(x) = x^5$

(c) $g(v) = e^v, \;\; u(x) = x^2 + 1$

(d) $g(v) = \ln(v), \;\; u(x) = 3x^2 + 4$

Composite Functions

Example

Find $y = f(x) = g(u(x))$ if

(a) $g(v) = v^5, \;\; u(x) = 2x + 1 \Longrightarrow f(x) = g(u(x)) = (2x + 1)^5$

(b) $g(v) = 2v + 1, \;\; u(x) = x^5 \Longrightarrow f(x) = g(u(x)) = 2x^5 + 1$

(c) $g(v) = e^v, \;\; u(x) = x^2 + 1 \Longrightarrow f(x) = g(u(x)) = e^{x^2+1}$

(d) $g(v) = \ln(v), \;\; u(x) = 3x^2 + 4 \Longrightarrow f(x) = g(u(x)) = \ln(3x^2 + 4)$

They are all functions of x.

Rewrite functions as Composite Functions

The following are Simple functions, and their derivatives are known:

- x^a
- e^x, a^x
- $\ln x$, $\log_a x$

Example

Write each function as a composition of simpler functions.

(a) $y = (3x^2 - x + 5)^4$

(b) $y = e^{x^4+2x^2+5}$

(c) $y = \ln\left(1 - x^2 + 2x^4\right)$

(d) $y = \left[\ln\left(x^2 + 3\right)\right]^{3/2}$

Rewrite functions as Composite Functions

Simple functions: x^a, e^x, a^x, $\ln x$, $\log_a x$

Example

Write each function as a composition of simpler functions.

(a) $y = (3x^2 - x + 5)^4 \Longrightarrow y = u^4, u = u(x) = 3x^2 - x + 5$

(b) $y = e^{x^4+2x^2+5} \Longrightarrow y = e^u, u = x^4 + 2x^2 + 5$

(c) $y = \ln\left(1 - x^2 + 2x^4\right) \Longrightarrow y = \ln u, u = 1 - x^2 + 2x^4$

(d) $y = \left[\ln\left(x^2+3\right)\right]^{3/2} \Longrightarrow y = u^{3/2}, u = \ln v, v = x^2 + 3$

Chain Rule

The following is the chain rule, which includes all the previous theorems in this lecture:

Theorem

(Chain Rule) Suppose $y = f(x) = g(u(x))$. Then

$$y' = \frac{dy}{dx} = f'(x) = g'(u(x))u'(x).$$

Chain Rule

Example

Find the derivative of $f(x) = (2x+1)^{100}$.

Solution

Let $u(x) = 2x + 1$ and $g(u) = u^{100}$, then

$$g'(u) = 100u^{99}, u'(x) = 2.$$

By using chain rule we obtain

$$f'(x) = g'(u)u'(x) = 100u^{99} \cdot 2 = 200(2x+1)^{99}$$

Remark

Theoretically, we can expand $(2x+1)^{100}$ into a polynomial and differentiate it term by term. But it will be very complicated.

General Power Rule

Theorem

(General Power Rule) If $u(x)$ is a differentiable function and n is any real number, and

$$f(x) = [u(x)]^n.$$

Then,

$$f'(x) = n[u(x)]^{n-1}u'(x).$$

Remarks

- Roughly speaking, we differentiate the function like the standard power rule $(([\cdot]^n)' = n[\cdot]^{n-1})$ and then multiply it by the derviative of the expression inside the bracket.

General Power Rule

Example

Find the derivative for each of the following functions:

(a) $(3x+1)^4$

(b) $(x^3+4)^{11}$

(c) $\sqrt{x^2-1}$

(d) $\dfrac{1}{2x^2+3}$

Answers

(a) $12(3x+1)^3$; (b) $33x^2(x^3+4)^{10}$; (c) $\frac{x}{\sqrt{x^2-1}}$; (d) $-\frac{4x}{(2x^2+3)^2}$.

General Rule for Exponential Functions

Similar to the general power rule, we have the following theorem for exponential functions:

Theorem

If $u(x)$ is a differentiable function and $f(x) = e^{u(x)}$. Then,

$$f'(x) = e^{u(x)}u'(x).$$

Remarks

- Roughly speaking, we differentiate the function like a standard exponential function ($(e^{[\cdot]})' = e^{[\cdot]}$) and then multiply it by the derviative of the expression inside the bracket.
- If $u(x) = x$, then $u'(x) = 1$ and the formula becomes the standard differentiation rule for exponential functions.

General Rule for Exponential Functions

Example

Find the derivative for each of the following functions:

(a) e^{x^2}

(b) $3e^{\sqrt{3x+5}}$

Solutions

(a) $(e^{x^2})' = e^{x^2}(x^2)' = 2xe^{x^2}$

(b)

$$(3e^{\sqrt{3x+5}})' = 3e^{\sqrt{3x+5}}(\sqrt{3x+5})' = 3e^{\sqrt{3x+5}}\frac{1}{2}(3x+5)^{-\frac{1}{2}}\cdot 3$$

$$= \frac{9}{2}e^{\sqrt{3x+5}}(3x+5)^{-\frac{1}{2}}$$

General Rule for Logarithmic Functions

Similar to the general power rule, we have the following theorem for logarithmic functions:

Theorem

If $u(x)$ is a differentiable function and $f(x) = \ln(u(x))$ Then,

$$f'(x) = \frac{1}{u(x)}u'(x).$$

Remarks

- Roughly speaking, we differentiate the function like a standard logarithmic function ($(\ln[\cdot])' = \frac{1}{[\cdot]}$) and then multiply it by the derviative of the expression inside the bracket.
- If $u(x) = x$, then $u'(x) = 1$ and the formula becomes the standard differentiation rule for logarithmic function.

General Rule for Logarithmic Functions

Example

Find the derivative for each of the following functions:

(a) $\ln(1 + 2x^4)$

(b) $x \ln(4x^6 + x - 1)$

Solutions

(a) $(\ln(1 + 2x^4))' = \frac{1}{1 + 2x^4} \cdot (1 + 2x^4)' = \frac{8x^3}{1 + 2x^4}$

(b) $(x \ln(4x^6 + x - 1))' = \ln(4x^6 + x - 1) + x(\ln(4x^6 + x - 1))'$

$$= \ln(4x^6 + x - 1) + \frac{x}{4x^6 + x - 1} \cdot (4x^6 + x - 1)'$$

$$= \ln(4x^6 + x - 1) + \frac{x(24x^5 + 1)}{4x^6 + x - 1}$$

Brief Summary

Remarks

- When $g(u) = u^n$, then the chain rule becomes the general power rule.
- When $g(u) = e^u$, then the chain rule becomes the general rule for exponential functions.
- When $g(u) = \ln(u)$, then the chain rule becomes the general rule for logarithmic functions.

Using Various Differentiation Rules Together

Example

Find the derivative for each of the following functions:

(a) $\frac{2x}{\sqrt{x^2 + 1}}$

(b) $\sqrt{(3x - 1)^3(x^2 + 1)}$

(c) $e^{\sqrt{2x+5}}$

Solution

(a) $\frac{2}{(x^2+1)^{3/2}}$; (b) $\sqrt{\frac{3x-1}{x^2+1}} \cdot \frac{15x^2-2x+9}{2}$; (c) $\frac{e^{\sqrt{2x+5}}}{\sqrt{2x+5}}$.