MATH 1003 Calculus and Linear Algebra (Lecture 18)

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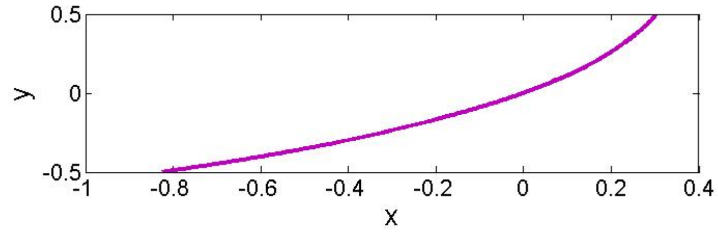
Implicit Functions

Draw the curve in x-y plane given by

$$F(x, y) = xe^y - y = 0,$$

where F is a function symbol with two inputs x and y.



In this case, y is essentially a function of x, which can not be explicitly expressed as $y = f(x)$, but is only defined implicitly by the equation $F(x, y) = 0$.

Implicit Differentiation

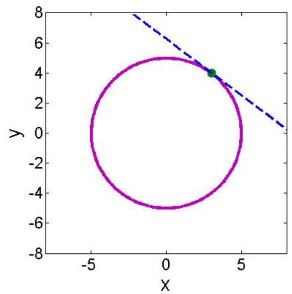
The slope of the tangent line to the curve in the previous slide exists, that is, we can still define dy/dx. This can be achieved by treating y both as a variable and a function symbol depending on x, i.e. $y = y(x)$.

Question

How can we find $\frac{dy}{dx} = y'(x)$ given $F(x, y) = c$, where c is a constant?

Example

Find the equation of line tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$ as shown in the figure on the right.



Implicit Differentiation - Example 1

Solution - Part 1

In this case, $F(x, y) = x^2 + y^2 = 25$. The slope of the tangent line at $(3, 4)$ is denoted by $\left.\frac{dy}{dx}\right|_{(3,4)}$, We differentiate both sides of $F(x, y) = x^2 + y^2 = 25$ with respect to x:

$$(x^2)' + \underbrace{(y^2)'}_{y \text{ is a function of } x:\ y(x)} = (25)'.$$

$(x^2)' = 2x$ and $(25)' = 0$. But $(y^2)' \neq 2y$. Hence, by general power rule (or more generally, chain rule), $(y^2)' = 2y\frac{dy}{dx}$. So the equation becomes

$$2x + 2y\frac{dy}{dx} = 0$$

Implicit Differentiation - Example 1

Solution - Part 2

Now, we obtain an equation that relates x, y and $\frac{dy}{dx}$. Therefore, we can express $\frac{dy}{dx}$ in terms of x and y as follows:

$$\frac{dy}{dx} = -\frac{x}{y}$$

By using $x = 3, y = 4$, we obtain

$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}$$

. Therefore, the equation of the tangent line is

$$\frac{y-4}{x-3} = -\frac{3}{4} \Rightarrow 4y - 16 = -3x + 9 \Rightarrow 3x + 4y - 25 = 0$$

Procedure to conduct implicit differentiation

To find the derivative of y with respect to x, denoted by $\left.\frac{dy}{dx}\right|_{(a,b)}$, we need

1. Find the formula $F(x, y) = c$
2. Differentiate $F(x, y) = c$ on both sides. It is important to treat y as a function of x, so that chain rule has to be applied for terms containing y.
3. Rearrange to express $\frac{dy}{dx}$ by x and y.
4. Evaluate $\frac{dy}{dx}$ at $x = a$ and $y = b$.

Implicit Differentiation - Example 2

Example

Find the equation(s) of line(s) tangent to the graph of

$$y - xy^2 + x^2 + 1 = 0$$

at the point(s) where $x = 1$.

Implicit Differentiation - Example 2

Solution - Part 1

In this case, $F(x, y) = y - xy^2 + x^2 + 1 = 0$ Differentiate both sides of the equation with respect to x, we have

$$(y)' - (xy^2)' + (x^2)' + (1)' = (0)'$$

We have $y' = \frac{dy}{dx}$, $(x^2)' = 2x$, $(1)' = 0$ and $(0)' = 0$. How about $(xy^2)'$? Since y is a function of x, xy^2 is a product of two functions of x, namely x and y^2. Therefore, we need to use product rule when differentiating:

$$(xy^2)' = (x)'y^2 + x(y^2)' = y^2 + x(2y\frac{dy}{dx})$$

Implicit Differentiation - Example 2

Solution - Part 2

Hence, the equation becomes

$$\frac{dy}{dx} - (y^2 + 2xy\frac{dy}{dx}) + 2x = 0$$

Group the terms with $\frac{dy}{dx}$ to the left hand side and the rest to the right hand side, we have

$$\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 - 2x$$

$$\Rightarrow (1 - 2xy)\frac{dy}{dx} = y^2 - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 2x}{1 - 2xy}$$

Implicit Differentiation - Example 2

Solution - Part 3

Now we need to find the point(s) on the graph with $x = 1$. Incorporating $x = 1$ into the equation gives

$$y - y^2 + 2 = 0 \Rightarrow (-y + 2)(y + 1) = 0 \Rightarrow y = 2, -1$$

Therefore, $(1, 2)$ and $(1, -1)$ are the required points on the graph and

$$\left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{2}{3}, \quad \left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{1}{3}$$

Similar to the previous example, we can use the above information to find the equations of two tangent lines (exercise).

Procedure to conduct implicit differentiation

To find the derivative of y with respect to x, denoted by $\left.\frac{dy}{dx}\right|_{(a,b)}$, we need

1. Find the formula $F(x, y) = c$
2. Differentiate $F(x, y) = c$ on both sides. It is important to treat y as a function of x, so that chain rule has to be applied for terms containing y.
3. Rearrange to express $\frac{dy}{dx}$ by x and y.
4. If the value of y is unknown, then calculate it by solving $F(a, y) = c$.
5. Evaluate $\frac{dy}{dx}$ at $x = a$ and $y = b$.

Implicit Differentiation - Exercises

Example

Find $\frac{dy}{dx}$ and evaluate $\frac{dy}{dx}$ at the indicated point.

(a) $x \ln y = ye^x - 1$ at $x = 0$.

(b) $e^{xy} - 2x = y + 1$ at $x = -1/2$.

Answers

(a) $\left.\frac{dy}{dx}\right|_{(0,1)} = \left.\frac{\ln y - ye^x}{e^x - x/y}\right|_{(0,1)} = -1.$

(b) $\left.\frac{dy}{dx}\right|_{(-1/2,0)} = \left.\frac{2 - ye^{xy}}{xe^{xy} - 1}\right|_{(-1/2,0)} = -4/3.$