MATH 1003 Calculus and Linear Algebra (Lecture 19)

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Related Rates

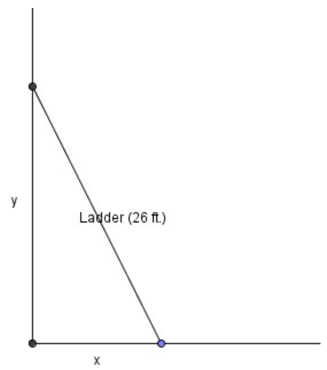
Given two quantities x and y, which are related by an equation. Suppose these quantities are also dependent on time t i.e. $x = x(t)$ and $y = y(t)$. Then from the given equation we can derive another equation which involves $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These are called the related rates.

The equation about the related rates can be obtained by differentiating both sides of the given equation with respect to t. The technique is very similar to implicit differentiation.

Related Rates - Example 1

Example

A 26-foot ladder is placed against a vertical wall. If the top of the ladder is sliding down the wall at 2 feet per second, at which rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet from the wall?



Related Rates - Example 1

Solution Part 1

First, we let x be the distance from the wall to the bottom of the ladder and y be the distance from the floor to the top of the ladder.

- independent variable: t
- dependent variable: $x(t)$ and $y(t)$
- restriction: by Pythagoras Theorem, we have

$$x^2 + y^2 = 26^2$$

We differentiate both sides of the equation with respect to t, regarding x and y as functions of t. Therefore, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solution Part 2

According to the given information, $\frac{dy}{dt} = -2$ when $x = 10$ (the rate is negative because the ladder is sliding down i.e. y is decreasing). To find $\frac{dx}{dt}$, we also need y when $x = 10$:

$$10^2 + y^2 = 26^2 \Rightarrow y = 24$$

Putting all the known values into the equation about related rates, we have

$$2 \cdot 10 \cdot \frac{dx}{dt} + 2 \cdot 24 \cdot (-2) = 0 \Rightarrow \frac{dx}{dt} = 4.8 \text{ ft/s}$$

Example

Suppose that for a company manufacturing transistor radios, the cost, revenue, and the profit equations are given by

$$C(x) = 5,000 + 2x, \quad R(x) = 10x - 0.001x^2, \quad P(x) = R(x) - C(x),$$

where the production output in 1 week is x radios. If production is increasing at the rate of 500 radios per week when producing is 2,000 radios, find the rates of increasing in cost, revenue and profit? (i.e. dC/dt, dR/dt and dP/dt?)

Solution

It is suggested that when $x = 2000$, $\frac{dx}{dt} = 500$. Thus differentiating C with respect to t gives

$$\frac{dC}{dt} = 2\frac{dx}{dt} = 2 \times 500 = 1000.$$

Differentiating R with respect to t, we have

$$\frac{dR}{dt} = 10\frac{dx}{dt} - 0.002x\frac{dx}{dt} = 10 \times 500 - 0.002 \times 2000 \times 500 = 3000.$$

Differentiating P with respect to t, we have

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt} = 3000 - 1000 = 2000$$

What does the solution imply? Ch.11-7 for your interest.

Example

The surface area of a spherical balloon is increasing at the rate of 4 cm^2 per minute.

(a) How fast is the radius changing when the surface area is 20 cm^2?

(b) Find the rate of change of volume of the balloon when the radius is 5 cm.

(Formulas: Surface area of a sphere with radius $S = 4\pi r^2$; Volume of a sphere with radius $V = \frac{4}{3}\pi r^3$.)

Related Rates - Example 3

Solution for (a)

Let V and S be the volume and the surface area of the balloon respectively.

By the given formula, $S = 4\pi r^2$. Differentiating it with respect to t gives

$$\frac{dS}{dt} = 4\pi \frac{d(r^2)}{dt} = 8\pi r \frac{dr}{dt}.$$

When $S = 20$ and $\frac{dS}{dt} = 4$, we also need the value for r from $r = \sqrt{S/4\pi} = \sqrt{5/\pi}$. Hence we have

$$\frac{dr}{dt} = \frac{1}{8\pi\sqrt{5/\pi}} \cdot \frac{dS}{dt} = \frac{1}{2\sqrt{5\pi}}.$$

Related Rates - Example 3

Solution for (b)

By the given formula, $V = \frac{4}{3}\pi r^3$. Differentiate it with respect to t, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Also, we differentiate the formula $S = 4\pi r^2$ with respect to t,

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Combine them we get

$$\frac{dV}{dt} = \frac{1}{2} r \frac{dS}{dt}$$

When $r = 5$, $\frac{dS}{dt} = 4$, we have $\frac{dV}{dt} = \frac{1}{2} \cdot 5 \cdot 4 = 10$.

Summary

The following is the procedure for solve a related rates problem:

1. Identify variable structure, i.e. find who is independent (t in this lecture), who depends on whom, etc.
2. Express all given rates and rates to be found as derivatives of dependent variables with respect to t.
3. Find an equation connecting the quantities identified in step 2.
4. Implicitly differentiate the equation with respect to t in step 3.
5. Solve for the derivative that will give the unknown rate.

One More Example

Example

Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, what is the rate of change of the distance between them after 2 hours? (Example 2 in Ch.11-6 of the textbook)