MATH 1003 Calculus and Linear Algebra (Lecture 20)

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Increasing and Decreasing Functions

Definition

Suppose $y = f(x)$.

1. $f(x)$ is increasing on an interval $a < x < b$, if for any $a < x_1 < x_2 < b$, $f(x_1) < f(x_2)$.
2. $f(x)$ is decreasing on an interval $a < x < b$, if for any $a < x_1 < x_2 < b$, $f(x_2) < f(x_1)$.

Increasing and Decreasing Functions

Example

Determine the interval for which the function $f(x) = x^2$ is increasing.

Solution

- From the graph of $y = x^2$, it is obvious that the function is increasing when $x > 0$ and decreasing when $x < 0$.
- Notice that the slope of the tangent is positive when f is increasing, and is negative when f is decreasing.

Increasing and Decreasing Functions

Theorem

For the interval $a < x < b$,

1. *$f(x)$ is an increasing function on $a < x < b$ if $f'(x) > 0$ on $a < x < b$.*
2. *$f(x)$ is a decreasing function on $a < x < b$ if $f'(x) < 0$ on $a < x < b$.*

Increasing and Decreasing Functions

Example

Given the function $f(x) = 8x - x^2$,

(a) Which value(s) of x correspond to horizontal tangent line?

(b) For which values of x is $f(x)$ increasing? Decreasing?

Solution

$f'(x) = 8 - 2x$. Therefore, $f'(x) = 0$ implies $x = 4$ i.e. The tangent line is horizontal at $x = 4$.

By the theorem, f is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. Therefore,

- f is increasing when $x < 4$.
- f is decreasing when $x > 4$.

Critical Numbers

Definition

The values of x in the domain of f where

1. $f'(x) = 0$, or
2. $f'(x)$ does not exist

are called the critical numbers of f.

Remark

The critical number(s) is(are) the point(s) that partition the number line into intervals, on which $f'(x)$ is either positive or negative i.e. f is either increasing or decreasing.

Critical Numbers

Example

Find the critical number(s) for the following f, and determine the interval(s) where f is increasing and those where f is decreasing.

(a) $f(x) = 1 - x^3$

(b) $f(x) = (1 + x)^{1/3}$

(c) $f(x) = 8 \ln x - x^2$

Critical Numbers

Solution

(a) $f'(x) = -3x^2$

$$f'(x) = 0 \Rightarrow x = 0$$

$x = 0$ is the critical number, and $f(x)$ is decreasing everywhere.

(b) $f'(x) = (1 + x)^{-2/3}/3$

$$f'(x) \text{ does not exist} \Rightarrow x = -1$$

$x = -1$ is the critical number, and $f(x)$ is an increasing function for all x.

Critical Numbers

Solution

(c) The domain of $f(x)$ is $x > 0$, and $f'(x) = 8/x - 2x$.

$$f'(x) = 0 \Rightarrow x = 2,$$

$f'(x)$ does not exist $\Rightarrow x = 0$(but not in the domain of $f(x)$.

Thus $x = 2$ is the critical number and $f(x)$ is increasing between 0 and 2 and decreasing when $x > 2$.

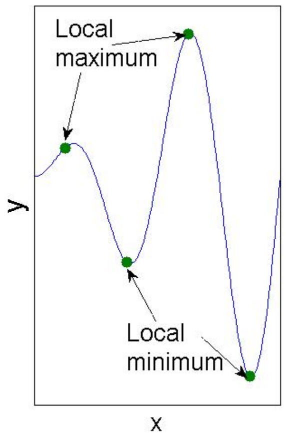
Local Extrema

Definition

Given a function $f(x)$,

1. We call $f(c)$ is a local maximum if there exists an interval $a < x < b$ containing c such that $f(x) \le f(c)$ for all x in $a < x < b$.
2. We call $f(c)$ is a local minimum if there exists an interval $a < x < b$ containing c such that $f(c) \le f(x)$ for all x in $a < x < b$.

The quantity $f(c)$ is called a local extrema if it is either a local maximum or local minimum.



Local Extrema

Theorem

If f is continuous on the interval $a < x < b$, c is a number in $a < x < b$ and $f(c)$ is a local extremum, then either

1. *$f'(c) = 0$, or*
2. *$f'(c)$ does not exist.*

That is, a local extremum can occur only at a critical value.

Remark

The theorem does not imply that every critical value produces a local extremum! (Think of $f(x) = x^3$ at $x = 0$.)

First-Derivative Test

We now present a method to classify whether a critical value is a local extrema:

1. $f(c)$ is a local minimum if $f'(x)$ changes from negative to positive at c. $(- \cdot +)$
2. $f(c)$ is a local maximum if $f'(x)$ changes from positive to negative at c. $(+ \cdot -)$
3. $f(c)$ is not a local extrema if $f'(x)$ does not change the sign at c. $(+ \cdot +)$ or $(- \cdot -)$

This is called the first-derivative test.

First-Derivative Test

Example

Given $f(x) = x^3 - 9x^2 + 24x - 10$.

(a) Find the critical value(s) of f.

(b) Find the local maxima and minima.

Solution for (a)

$f'(x) = 3x^2 - 18x + 24$. Obviously, $f'(x)$ exists for all values of x. Therefore, the critical values occur when

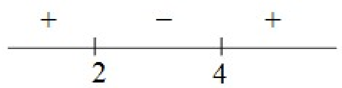
$$3x^2 - 18x + 24 = 3(x-2)(x-4) = 0.$$

The critical values are 2 and 4.

First-Derivative Test

Solution for (b)

Using the critical values obtained in (a), we can construct the sign chart for $f'(x)$:



f is increasing when $x < 2$ or $x > 4$ and is decreasing when $2 < x < 4$. Therefore, we have the following results:

- f has a local maximum at $x = 2$.
- f has a local minimum at $x = 4$.

First-Derivative Test

Example

Find the critical values, the interval(s) where $f(x)$ is increasing, the interval(s) where $f(x)$ is decreasing, and the local extrema of the following functions

(a) $f(x) = \dfrac{9}{x} + x$

(b) $f(x) = \dfrac{x^2}{x+1}$

(c) $f(x) = 3(x-2)^{2/3} + 2$.

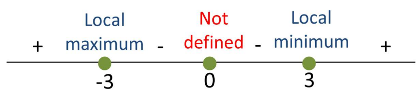
(d) $f(x) = (x-1)e^{-x}$

Exercises

Solution

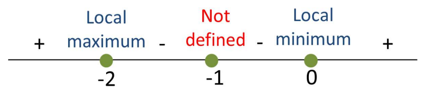
(a)

$$f'(x) = 1 - 9/x^2.$$



(b)

$$f'(x) = \frac{x^2 + 2x}{(x+1)^2}.$$



(c)

$$f'(x) = 2(x-2)^{-1/3}.$$

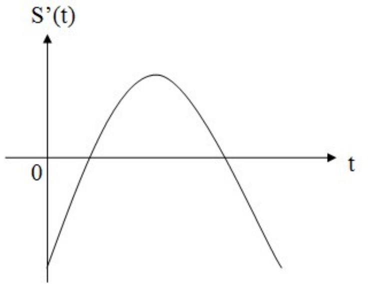
(d)

$$f'(x) = (2-x)e^{-x}.$$

Exercises

Example

The graph in below approximates the rate of change of the U.S. share of the total world production of motor vehicles over a 20-years period, where $S(t)$ is the U.S. share and t is time.



- Find all the local extrema (if any).