MATH 1003 Calculus and Linear Algebra (Lecture 21)

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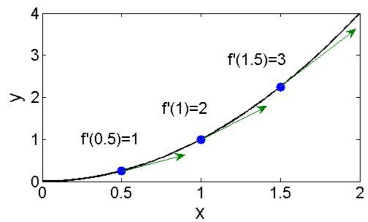
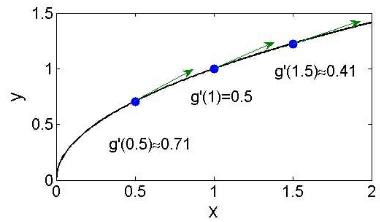
Concavity

Example

$f(x) = x^2$ $\qquad$ $g(x) = \sqrt{x}$





$f'(x)$ and $g'(x)$ increasing or decreasing?

x	0.5	1	1.5
$f'(x)$	1	2	3
$g'(x)$	0.71	0.5	0.41

Concavity

Definition

Concavity of $f(x)$:

1. The graph of a function is concave upward on the interval $a < x < b$ if $f'(x)$ is increasing on $a < x < b$.
2. The graph of a function is concave downward on the interval $a < x < b$ if $f'(x)$ is decreasing on $a < x < b$.

Second Derivatives

Definition

For $y = f(x)$, the second derivative of f, is

$$\frac{d^2y}{dx^2}(x) = f''(x) = \frac{d}{dx}f'(x).$$

Remark

Similarly, we can define the n^{th} derivative of f by differentiating f n times. The notation for the n^{th} derivative is

$$\frac{d^ny}{dx^n}(x)$$

Second Derivatives

Example

Find the second derivatives of

(a) $f(x) = \sqrt[3]{x^2} + 3x^3 + \dfrac{1}{x^7}$

(b) $y = x^2 \log_5 x$

Second Derivatives

Solution

(a) $f'(x) = \frac{2}{3}x^{-1/3} + 9x^2 - 7x^{-8}$

$\Rightarrow f''(x) = (f'(x))' = -\frac{2}{9}x^{-4/3} + 18x + 56x^{-9}$

(b) $\frac{dy}{dx} = 2x\log_5 x + \frac{x^2}{x\ln 5} = 2x\log_5 x + \frac{x}{\ln 5}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = 2\log_5 x + \frac{2}{\ln 5} + \frac{1}{\ln 5} = 2\log_5 x + \frac{3}{\ln 5}.$$

Concavity and Second Derivatives

By the definition of concavity, we can easily obtain the following theorem:

Theorem

Suppose f is twice differentiable on an interval $a < x < b$. Then

1. *$f(x)$ is a concave upward on $a < x < b$ if and only if $f''(x) > 0$ on $a < x < b$, and*
2. *$f(x)$ is a concave downward on $a < x < b$ if and only if $f''(x) < 0$ on $a < x < b$.*

Concavity and Second Derivatives

Example

Determine the interval(s) where the graph of f is concave upward and the interval(s) where the graph of f is concave downward.

(a) $f(x) = x^3$

(b) $g(x) = \ln x$

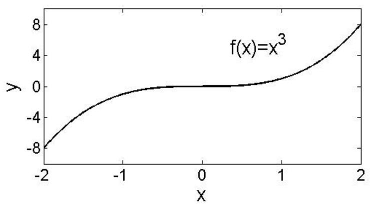
(c) $h(x) = \sqrt{x+1}$

(d) $k(x) = x^{\frac{4}{3}}$

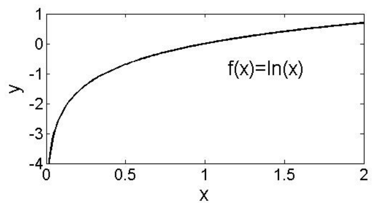
Concavity and Second Derivatives

Solutions

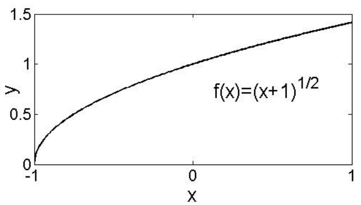
(a) $f'(x) = 3x^2 \Rightarrow f''(x) = 6x$



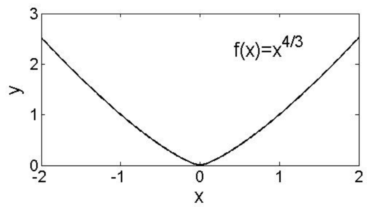
(b) $g'(x) = \frac{1}{x} \Rightarrow g''(x) = -\frac{1}{x^2}$



(c) $h'(x) = \frac{1}{2\sqrt{x+1}} \Rightarrow$
$h''(x) = -(x+1)^{-3/2}/4$



(d) $k'(x) = \frac{4}{3}x^{1/3} \Rightarrow$
$k''(x) = 4x^{-2/3}/9$



Inflection Points

Definition

An inflection point is a point on the graph of the function where the concavity changes.

To locate an inflection point, we have the following theorem:

Theorem

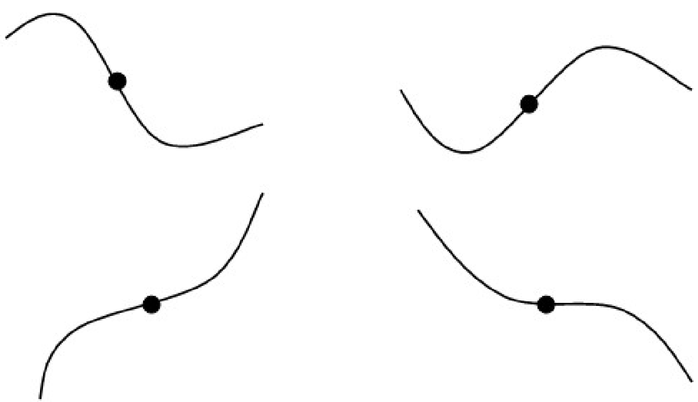
If $y = f(x)$ has an inflection point point at $x = c$, then

1. $f''(c) = 0$, *or*
2. $f''(c)$ *does not exist.*

Remark

The converse of the above the theorem is not true.

Examples of Inflection Points



Examples of Inflection Points

Example

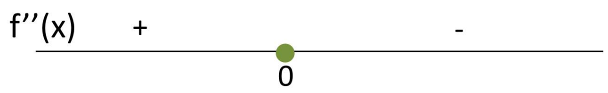
Find all inflection point(s) of $f(x)$ (if any):

(a) $f(x) = x^{\frac{1}{3}}$

(b) $f(x) = -x^{-3}$

Solution (a)

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$. Hence the sign chart for $f''(x)$ is as follows:

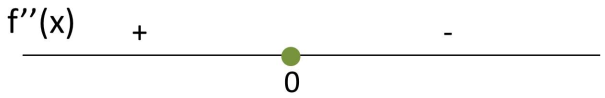


Since there is a sign change in $f''(x)$, $x = 0$ is an inflection point.

Examples of Inflection Points

Solution (b)

$f''(x) = -12x^{-5}$. Hence the sign chart for $f''(x)$ is as follows:



Although there is a sign change in $f''(x)$, $x = 0$ is NOT an inflection point! Why?

Notice that $f(x)$ is undefined at $x = 0$. Therefore, the graph does not have any point at $x = 0$.

Examples of Inflection Points

Example

Determine the interval(s) where the graph of f is concave upward and the interval(s) where the graph of f is concave downward. Indicate all inflection point(s) (if any).

(a) $f(x) = x^3 - 9x^2 + 24x - 10$.

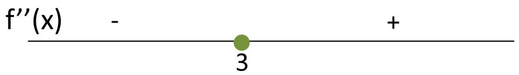
(b) $f(x) = \ln(x^2 - 4x + 5)$

Examples of Inflection Points

Solutions

(a) $f'(x) = 3x^2 - 18x + 24$

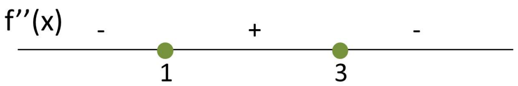
$f''(x) = 6x - 18 \Rightarrow$ inflection point: $x = 3$.



$x > 3$, upwards; $x < 3$, downwards.

(b) $f'(x) = \frac{2x-4}{x^2-4x+5}$

$f''(x) = \frac{-2(x-1)(x-3)}{(x^2-4x+5)^2} \Rightarrow$ inflection point: $x = 1, 3$.



$1 < x < 3$, upwards; $x < 1$ or $x > 3$, downwards.