MATH 1003 Calculus and Linear Algebra (Lecture 22)

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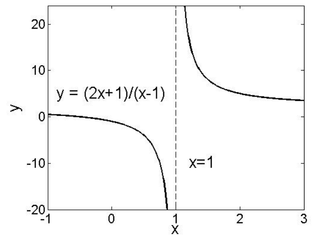
Asymptotes of Fraction of Linear Functions

Given $f(x) = \frac{a_0 x + a_1}{b_0 x + b_1}$, $b_0 \neq 0$

$$\lim_{x \to \pm\infty} f(x) = \frac{a_0}{b_0};$$

$$\lim_{x \to (-b_1/b_0)} = \infty,$$

where a is either $-$ or $+$ to be decided.



For the graph $f(x) = \frac{a_0 x + a_1}{b_0 x + b_1}$,

- the line $y = \frac{a_0}{b_0}$ is called the horizontal asymptote,
- the line $x = -\frac{b_1}{b_0}$ is called the vertical asymptote.

Asymptotes of Exponential Functions with Base a

Given $f(x) = a^x$ with $a > 0$
If $a > 1$,

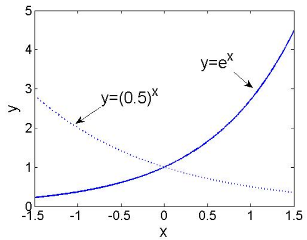
$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = 0;$$

if $0 < a < 1$,

$$\lim_{x \to +\infty} f(x) = 0$$

$$\lim_{x \to -\infty} f(x) = +\infty.$$



Asymptotes of Logarithmic Functions with Base a

Given $f(x) = \log_a x$ with $a > 0$ for $x > 0$
If $a > 1$,

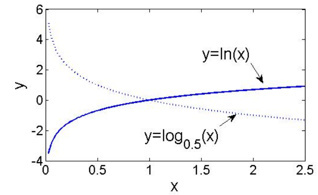
$$\lim_{x \to +\infty} f(x) = +\infty,$$

$$\lim_{x \to 0^+} f(x) = -\infty;$$

if $0 < a < 1$,

$$\lim_{x \to +\infty} f(x) = -\infty$$

$$\lim_{x \to 0^+} f(x) = +\infty.$$

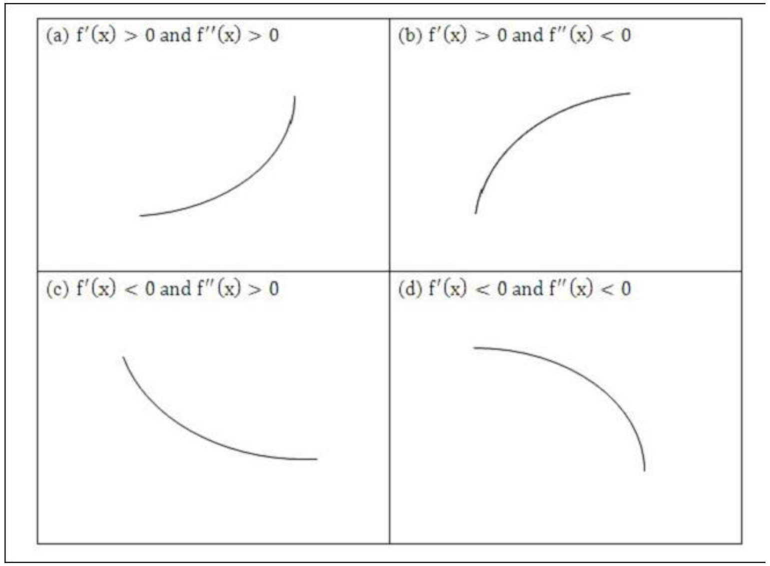


Curve Sketching - Procedure

To sketch the graph of $y = f(x)$:

1. Find the domain of $f(x)$.
2. Find asymptotes (vertical/horizontal), determine which direction the curve tends to $(\pm\infty)$.
3. Find the intercepts:
   - x-intercept: find x with $f(x) = 0$.
   - y-intercept: find $f(0)$.
4. Find $f'(x)$ and construct the sign chart for $f'(x)$. Then locate the critical numbers, local maxima, local minimum and intervals for which the function is increasing and decreasing.
5. Find $f''(x)$ and construct the sign chart for $f''(x)$. Then locate the inflection points and intervals for which the function is concave up and concave down.
6. Sketch the graph of $y = f(x)$.

General patterns of analyzing graphs



Example 0

Graph the function $y = f(x)$, if $f(x)$ satisfies the following:

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- $f(0) = 2$, $f(1) = 0$, $f(2) = -2$;
- $f'(0) = f'(2) = 0$; $f'(x) > 0$ on $-\infty < x < 0$ and $2 < x < \infty$; $f'(x) < 0$ on $0 < x < 2$;

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- $f(0) = 2$, $f(1) = 0$, $f(2) = -2$;
- $f'(0) = f'(2) = 0$; $f'(x) > 0$ on $-\infty < x < 0$ and $2 < x < \infty$; $f'(x) < 0$ on $0 < x < 2$;
- $f''(1) = 0$; $f''(x) > 0$ on $1 < x < \infty$; $f''(x) < 0$ on $-\infty < x < 1$.

Example 1

Graph $y = x^4 + 4x^3$.

Solution

- the function is defined everywhere.

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- y-intercept: Put $x = 0$, we get $y = 0$. Therefore, y-intercept is 0.
- x-intercept: Put $y = 0$, we get the equation $x^4 + 4x^3 = 0$. Factorizing, we have

$$x^3(x+4) = 0$$

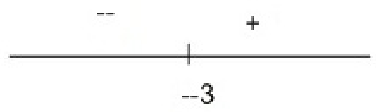
Hence $x = 0, -4$. Therefore, x-intercepts are 0 and -4.

Solution (cont'd)

$f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$.

- $f'(x) = 0$, the critical values are $x = 0, -3$.

The sign chart for $f'(x)$ is as follows:



Therefore, we have

- f is decreasing when $x < -3$.

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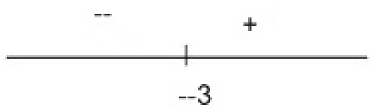

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- f is decreasing when $x < -3$.
- f is increasing when $x > -3$.

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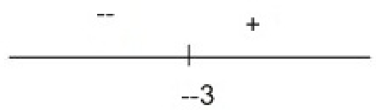

Therefore, we have

- f is decreasing when $x < -3$.
- f is increasing when $x > -3$.
- Local minimum occurs when $x = -3$.

Solution (cont'd)

$f''(x) = 12x^2 + 24x = 12x(x+2)$. The sign chart for $f''(x)$ is as follows:

Therefore, we have

- f is concave upwards when $x < -2$ or $x > 0$.

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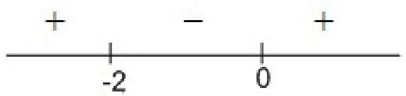
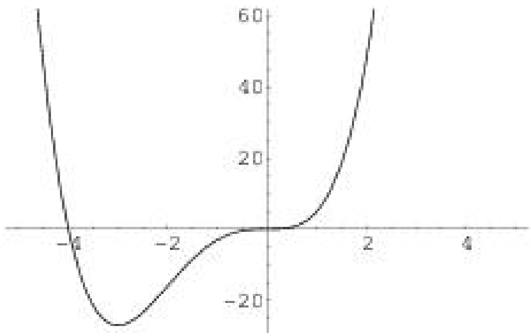


Therefore, we have

- f is concave upwards when $x < -2$ or $x > 0$.
- f is concave downwards when $-2 < x < 0$.
- $x = -2, 0$ are inflection points.

Graphing $y = f(x)$

- Plot the x-intercepts and y-intercept.
- Plot all the local extrema and inflection points.
- Connect the points by curves that are in accordance with the analysis.



Example 2

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- Domain of $f(x)$: $x > 0$.
- y-intercept: $x = 0$, not in the domain.
- x-intercept(s): Put $y = 0$, we get $x(\ln x)^2 = 0$, which implies that $x = 0$ (excluded) or $\ln x = 0 \Longrightarrow x = 1$. Hence x-intercept is 1.

Solution (cont'd)

Differentiate $f(x)$, we get $f'(x) = (\ln x)^2 + 2\ln x = (\ln x)(\ln x + 2)$. $f'(x) = 0$ then $\ln x = 0 \Longrightarrow x = 1$ or $\ln x = -2 \Longrightarrow x = e^{-2}$. The critical numbers are $x = 1, e^{-2} \approx 0.1353$. The sign chart for $f'(x)$ is as follows:

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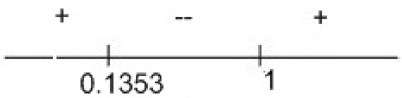

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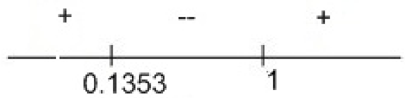

Therefore, $x = e^{-2}$ is local maximum and $x = 1$ is local minimum, $f(x)$ is increasing when $0 < x < e^{-2}$ or $x > 1$, and $f(x)$ is decreasing when $e^{-2} < x < 1$.

Solution (cont'd)

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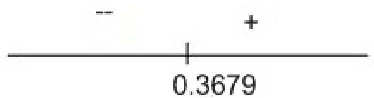
Differentiate $f'(x)$, we get

$$f''(x) = 2\ln x \frac{1}{x} + \frac{2}{x} = \frac{2}{x}(\ln x + 1).$$

Since $x > 0$, $f''(x) = 0$ if $\ln x + 1 = 0$, that is $x = e^{-1} = 0.3679$, the sign chart for $f''(x)$ is as follows:



Therefore, we have

- $f(x)$ is concave upwards when $x > e^{-1}$.

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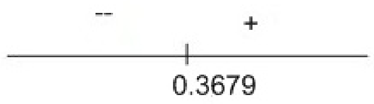

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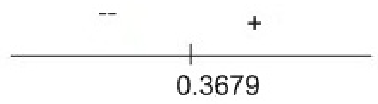


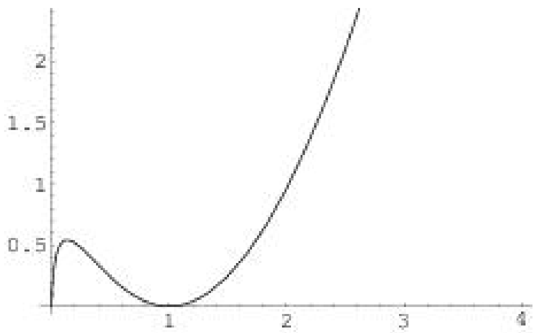
Therefore, we have

- $f(x)$ is concave upwards when $x > e^{-1}$.
- $f(x)$ is concave downwards when $0 < x < e^{-1}$.
- $x = e^{-1}$ is an an inflection point.

Solution (cont'd)

To sketch the graph of $y = f(x)$:

- Plot the x-intercept(s) and y-intercept.
- Draw the vertical and horizontal asymptote(s) using dotted lines.
- Plot all local extrema and inflection point(s) (if any).
- Connect the points by curves that are in accordance with the analysis.



Example 3

Example

Sketch $y = f(x) = \frac{2x-1}{x+1}$, where $x \in (-\infty, \infty)$ except for $x = -1$.

Solution - Part 1

- $f(x)$ at infinity behaves like $\lim_{x\to\pm\infty} f(x) = 2$.

Example 3

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Sketch $y = f(x) = \frac{2x-1}{x+1}$, where $x \in (-\infty, \infty)$ except for $x = -1$.

Solution - Part 1

- $f(x)$ at infinity behaves like $\lim_{x\to\pm\infty} f(x) = 2$.
- At $x = -1$, $f(x)$ is not defined. We find

$$\lim_{x\to -1^+} f(x) = -\infty, \qquad \lim_{x\to -1^-} f(x) = +\infty.$$

Example 3

Solution - Part 2

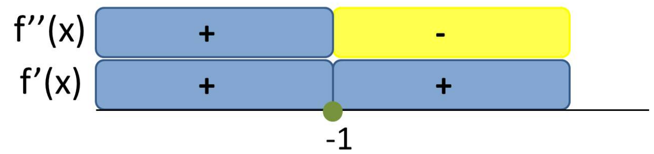
Since

$$\frac{dy}{dx} = f'(x) = \frac{3}{(x+1)^2} > 0, \text{ except for } x = -1,$$

but $f(x)$ is not defined at $x = -1$, so there is no critical points.

Since

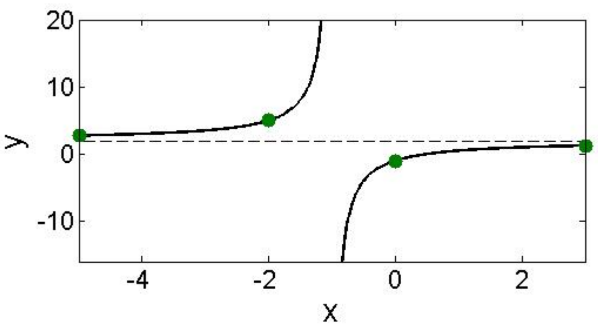
$$\frac{d^2y}{dx^2} = f''(x) = -\frac{6}{(x+1)^3} \Rightarrow \text{no inflection points.}$$



Example 3

Solution - Part 3

x	$-\infty$	-2	0	∞
$f(x)$	2	5	-1	2



Curve Sketching - Procedure

To sketch the graph of $y = f(x)$:

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   - x-intercept: find x with $f(x) = 0$.
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6. Sketch the graph of $y = f(x)$.

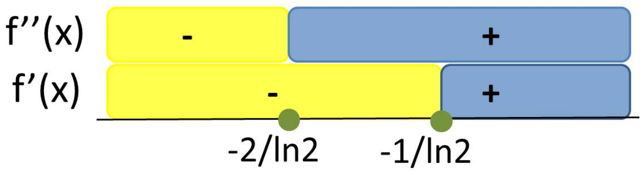
Example 4

Example

Sketch $y = f(x) = x2^x$, where $x \in [-8, 1]$.

Solution - Part 1

1. x lies between -8 and 1. At two boundaries, $f(-8) = -0.0313$ and $f(1) = 2$.
2. $f(x)$ is well-defined in $[-8, 1]$, no asymptotes.
3. $f'(x) = 2^x(1 + x\ln 2) \Rightarrow$ critical points: $x = -1/\ln 2$.
4. $f''(x) = 2^x((\ln 2)^2 x + 2\ln 2) \Rightarrow$ inflection points: $x = -2/\ln 2$.

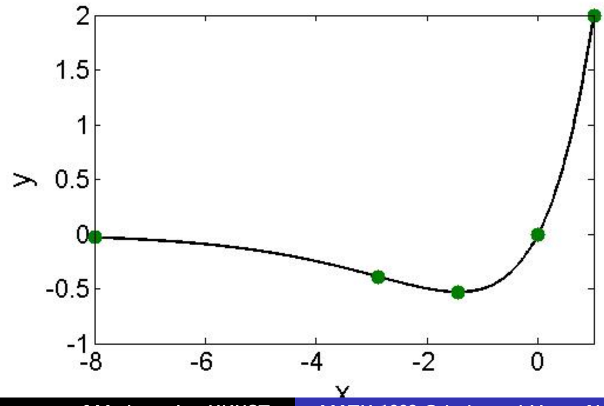


Example 4

Solution - Part 2

5. Evaluate $f(x)$ at all critical and inflection points:

x	-8	$-2/\ln 2$	$-1/\ln 2$	0	1
$f(x)$	-0.0313	-0.3905	-0.5307	0	2



Point of Diminishing Returns

The value of x where the rate of change of $f(x)$ changes from increasing to decreasing is called the point of diminishing returns.

Example

A discount appliance store is selling 200 large-screen TV sets monthly. If the store invests \$x thousand in an advertising campaign and the ad company estimates that sales will increase to

$$N(x) = 4x^3 - 0.25x^4 + 200, \quad 0 \le x \le 12.$$

When is rate of change of sales increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales?

Solution

$N'(x) = 12x^2 - x^3$ and $N''(x) = 24x - 3x^2 = 3x(8 - x)$. The sign chart for $N''(x)$ is as follows:

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Therefore, we have

- The rate of change of sales is increasing when $0 < x < 8$.

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- The rate of change of sales is increasing when $0 < x < 8$.
- The rate of change of sales is decreasing when $8 < x < 12$.

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$N'(x) = 12x^2 - x^3$ and $N''(x) = 24x - 3x^2 = 3x(8 - x)$. The sign chart for $N''(x)$ is as follows:

Therefore, we have

- The rate of change of sales is increasing when $0 < x < 8$.
- The rate of change of sales is decreasing when $8 < x < 12$.
- The point of diminishing returns is at $x = 8$.

Solution

$N'(x) = 12x^2 - x^3$ and $N''(x) = 24x - 3x^2 = 3x(8 - x)$. The sign chart for $N''(x)$ is as follows:

Therefore, we have

- The rate of change of sales is increasing when $0 < x < 8$.
- The rate of change of sales is decreasing when $8 < x < 12$.
- The point of diminishing returns is at $x = 8$.
- The maximum rate of change of sales occurs at $x = 8$. Therefore, the maximum rate is $N'(8) = 12 \times 8^2 - 8^3 = 256$.

Self-Test Problems

In textbook,

- Example 5 of Section 12-2
- Example 2 and 4 of Section 12-4