MATH 1003 Calculus and Linear Algebra (Lecture 23)

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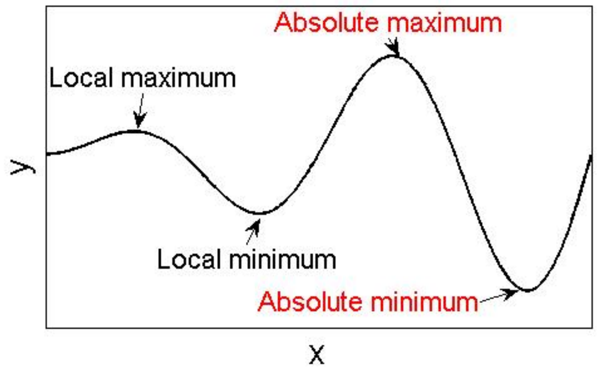
Local Extrema and Absolute Extrema

Definition

- If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the absolute maximum of f.
- If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the absolute minimum of f.

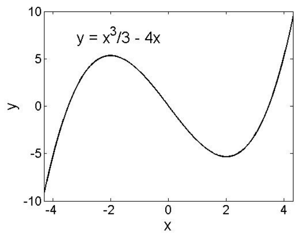


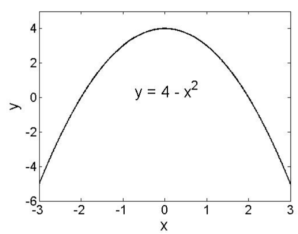
Local Extrema and Absolute Extrema

Remarks

- A local maximum/minimum may not be an absolute max/min.
- If f has only one critical value and it is a local max/min, then it is also an absolute max/min.
- There exist a function that has no absolute max/min.

Example



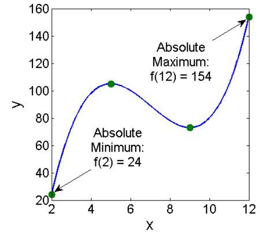


Local Extrema and Absolute Extrema

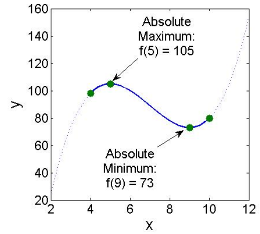
Example

The absolute extrema for $f(x) = x^3 - 21x^2 + 135x - 170$, when $x \in$
a) $[2, 12]$, b) $[4, 10]$, c) $[4, 8]$, d) $[3, 11]$.

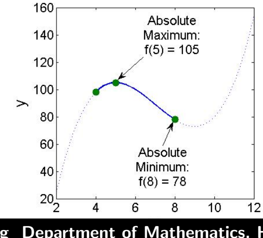
a)



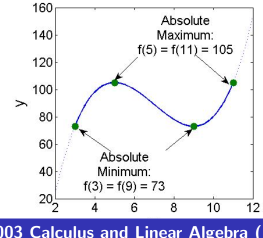
b)



c)



d)



A Theorem about Absolute Extrema

Theorem

A function that is continuous on an interval $[a, b]$ $(a \leq x \leq b)$ has both an absolute maximum value and an absolute minimum value on that interval. Moreover, the absolute extrema must always occur at

- *critical values, or*
- *the endpoints i.e a and b.*

Remark

The two assumptions (f is continous and the domain of f is a closed interval) in the theorem are very important. Without any one of them, the conclusion of the theorem may not hold. Consider the following examples:

- $f(x) = \frac{1}{x}$ on $(0, 1]$. f has no absolute maximum since $f(x) \to \infty$ as $x \to 0$.
- $f(x) = \frac{1}{(1-x)^2}$ on $[0, 2]$. f is discontinuous at $x = 1$ and f has no absolute maximum since $f(x) \to \infty$ as $x \to 1$.

Second Derivative Test

The following theorem is the second derivative test for absolute extrema:

Theorem

Let c be the only critical value of $f(x)$.

(a) *If $f''(c) > 0$, then $f(c)$ is a local minimum.*

(b) *If $f''(c) < 0$, then $f(c)$ is a local maximum.*

(c) *If $f''(c) = 0$, we have no conclusion.*

Second Derivative Test

Example

Find the absolute maximum and absolute minimum values of

$$f(x) = x^3 - 12x$$

on each of the following intervals:

(a) $[-5, 5]$

(b) $[-3, 3]$

(c) $[-3, 1]$

Find Absolute Extrema

Solution for (a)

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2) \Rightarrow \text{ critical values} : x = 2, -2.$$

$$f''(x) = 6x \Rightarrow \quad f''(2) = 12 > 0, f''(-2) = -12 < 0.$$

Hence at $x = 2$, $f(x)$ attains a local minimum; at $x = -2$, $f(x)$ attains a local maximum According to the previous theorem, absolute extrema must be at critical values and/or endpoints. We list out the values of f at critical values and endpoints to make comparison:

- For critical values: $f(2) = -16$, $f(-2) = 16$.
- For endpoints: $f(5) = 65$, $f(-5) = -65$.

Therefore, $f(-5) = -65$ is the absolute minimum and $f(5) = 65$ is the absolute maximum.

Find Absolute Extrema

Solution for (b)

When $x \in [-3, 3]$, we list out the values of f at critical values and endpoints to make comparison:

- For critical values: $f(2) = -16$, $f(-2) = 16$.
- For endpoints: $f(3) = -9$, $f(-3) = 9$.

Therefore, $f(2) = -16$ is the absolute minimum and $f(-2) = 16$ is the absolute maximum.

Solution for (c)

When $x \in [-3, 1]$, we list out the values of f at critical values and endpoints to make comparison:

- For critical values: $f(-2) = 16$ ($f(2)$ is not included).
- For endpoints: $f(-3) = 9$, $f(1) = -11$.

Therefore, $f(1) = -11$ is the absolute minimum and $f(-2) = 16$ is the absolute maximum.

Find Absolute Extrema

Example

Find the absolute minimum of each function on $0 < x < \infty$

(a) $f(x) = x + \dfrac{4}{x}$

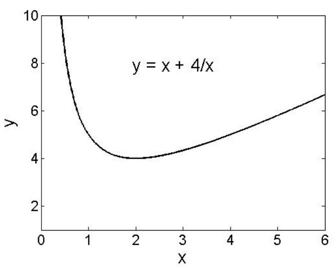
(b) $f(x) = (\ln x)^2 - 3 \ln x$

Find Absolute Extrema

Solution for (a)

- $f'(x) = 1 - 4/x^2 \Rightarrow$ critical values: $x = 2$.
- $f''(x) = 8/x^3 \Rightarrow f''(2) = 1 > 0$.

At $x = 2$, $f(x)$ attains a local minimum since it is the only local minimum, then it is an absolute minimum.



Solution for (b)

- $f'(x) = \frac{2\ln x - 3}{x} \Rightarrow$ critical values: when $\ln x = 3/2 \Rightarrow x = e^{3/2}$.
- $f''(x) = \frac{5 - 2\ln x}{x^2} \Rightarrow$ $f''(e^{3/2}) = \frac{2}{e^3} > 0$.

At $x = e^{3/2}$, $f(x)$ attains a local minimum since it is the only local minimum, then it is an absolute minimum.