MATH 1003 Calculus and Linear Algebra (Lecture 24)

Maosheng Xiong
Department of Mathematics, HKUST

Summary

Question
For a function $y = f(x)$ in a domain, how do we find the absolute maximum or minimum?

First Derivative Test

The following theorem is the first derivative test for local extrema:

Theorem
Let c be a critical number of $f(x)$. Then

Sign	$x < c$	$x > c$	$f(c)$ is
$f'(x)$	$-$	$+$	local min
$f'(x)$	$+$	$-$	local max
$f'(x)$	$-$	$-$	NA
$f'(x)$	$+$	$+$	NA

Second Derivative Test

The following theorem is the second derivative test for local extrema:

Theorem
Let c be a critical number of $f(x)$.

(a) *If $f''(c) > 0$, then $f(c)$ is a local minimum.*

(b) *If $f''(c) < 0$, then $f(c)$ is a local maximum.*

(c) *If $f''(c) = 0$, we have no conclusion.*

(a)   (b)

Absolute extrema

Theorem

If c is the *only* critical number of $f(x)$ in a domain. Then

(a) if $f(c)$ is a local minimum, then it is the absolute minimum.

(b) if $f(c)$ is a local maximum, then it is the the absolute maximum.

We can use either the first or the second derivative test to find local min/max.

A Theorem about Absolute Extrema

Theorem

A function that is *continuous* on an interval $[a, b]$ ($a \leq x \leq b$) has *both* an absolute maximum value and an absolute minimum value on that interval. Moreover, the absolute extrema must always occur at

▶ critical values, or

▶ the endpoints i.e a and b.

So in this case we can list all values $f(x)$ at the critical numbers and the endpoints and pick up the largest (absolute max) and the smallest (absolute min).

Example 1 - Maximizing Area

Example

A homeowner has \$320 to spend on building a fence around a rectangular garden. Three sides of the fence will be constructed with wire fencing at a cost of \$2 per linear foot. The fourth side will be constructed with wood fencing at a cost of \$6 per linear foot. Find the dimensions and the area of the largest garden that can be enclosed with \$320 worth of fencing.
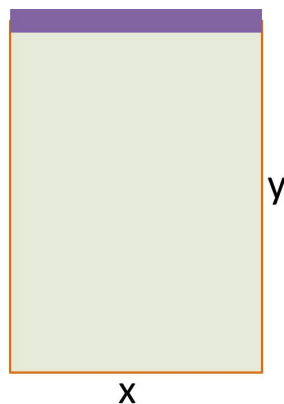
Example 1 - Maximizing Area

Solution - Part 1

Firstly, we define x and y as denoted in the figure. Hence the area A of the garden satisfies

$$A = xy.$$

The total cost C to build a rectangular garden of size $x \times y$ in this case is

$$C = 2y + 2x + 2y + 6x = 8x + 4y = 320.$$

Hence the problems becomes

To maximize $A = xy$, subject to $8x + 4y = 320$.

Example 1 - Maximizing Area

Solution - Part 2

We can express y as a function of x from the restriction:

$$8x + 4y = 320 \Rightarrow y = 80 - 2x.$$

Practically, both x and y are non-negative:

$$x \geq 0, \quad y = 80 - 2x \geq 0 \Rightarrow x \leq 40.$$

Hence the problems becomes

To maximize $A(x) = x(80 - 2x) = 80x - 2x^2$, where $0 \leq x \leq 40$.

Example 1 - Maximizing Area

Solution - Part 3

It can be calculated that

$$A'(x) = 80 - 4x \Rightarrow \text{ critical points: } x = 20.$$

The absolute maximum can only occur at

- ▶ critical values: $f(20) = 800$
- ▶ end points: $f(0) = f(40) = 0$.

Thus the maximum area that a garden can occupy in such fencing way is 800 square feet.
In this case, $y = 80 - 2 \times 20 = 40$, so the garden is of size 20×40.

Example 2 - Minimizing Perimeter

Example

Refer to Example 1. The owner judges that an area of 800 square feet for the garden is too small and decides to increase the area to 1,250 square feet. What is the minimum cost of building a fence that will enclose a garden with an area of 1,250 square feet? What are the dimensions of this garden? Assume that the cost of fencing remains unchanged.

1,250 square feet y

x
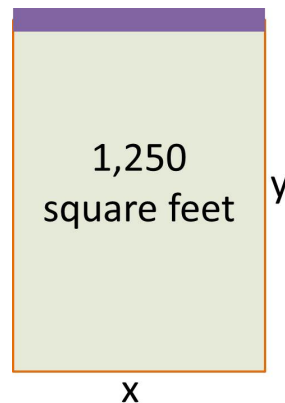
Example 2 - minimizing Perimeter

Solution - Part 1

This time, we want

$$\text{to minimize } C = 8x + 4y, \text{ subject to } A = xy = 1250.$$

To eliminate y, we have

$$y = \frac{A}{x}$$

Again in practice, both x and y are non-negative $\Rightarrow x > 0$. It is noted that there is no end points in this case. Hence the problems becomes

$$\text{to minimize} C(x) = 8x + 4 \cdot \frac{1250}{x} = 8x + \frac{5000}{x}, \text{ where } x > 0.$$

Example 2 - minimizing Perimeter

Solution - Part 2

It can be calculated that

$$\frac{dC}{dx} = C'(x) = 8 - \frac{5000}{x^2} \Rightarrow \text{ critical points: } x = \sqrt{\frac{5000}{8}} = 25.$$

Also we can calculate that $C''(x) = 10000/x^3$, this gives rise to $C(25) = 0.64 > 0$. Since $x = 25$ is the only critical value, $C(25)$ is the absolute minimum.

Therefore, when $x = 25$ and $y = 1250/25 = 50$, the garden costs least ($C = 8 \times 25 + 4 \times 50 = \$400$) to enclose an area of $1,250$ square feet.

Procedures for Optimisation Problems

1. Introduce variables, look for relationships among them.
2. Express the quantity to be maximized/minimized as a function of one free variable x, say $f(x)$. Find the interval I where $f(x)$ is defined (the interval I is often co-determined from the practical point of view). The mathematical model is built up as

   To maximize/minimize $f(x)$, subject to $x \in I$.

3. Find the absolute maximum / minimum from $f(x)$ evaluated at either a critical point or a end point.
4. Interpret the outcomes from the mathematical model.

Example 3 - Drug Concentration

Example

The concentration $C(t)$, in milligrams per cubic centimeter, of a particular drug in a patient's bloodstream is given by

$$C(t) = \frac{0.16t}{t^2 + 4t + 4},$$

where t is the number of hours after the drug is taken. How many hours after the drug is taken will the concentration be maximum? What is the maximum concentration?

Example 3 - Drug Concentration
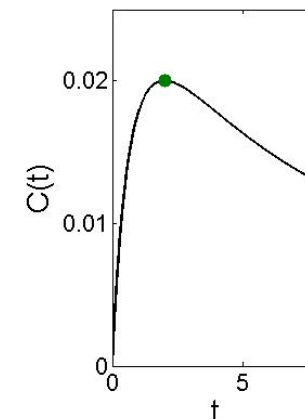
Solution

The mathematical model is that

To maximize $C(t)$, subject to $t > 0$.

It is calculated that

$$C'(t) = \frac{0.16(2 - t)}{(t + 2)^3}$$

$\Rightarrow$ critical point: $t = 2$.

It can be checked that $C''(t) = \frac{3t-7}{(t+2)^4}$, thus $C''(2) < 0$. Therefore, $C(t)$ attains the absolute maximum at the only critical point $t = 2$. The concentration then is $C(2) = 0.02$.
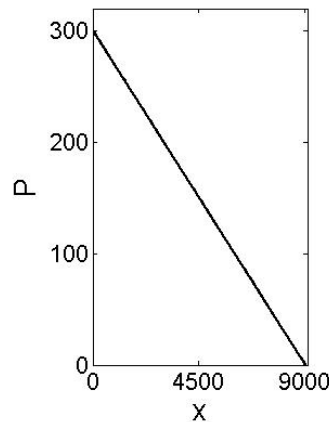
Example 4 - maximizing Revenue

Example

An office supply company sells heavy-duty paper shredders per year at P per shredder. The price-demand equation for these shredders is

$$P = 300 - \frac{x}{30}.$$

What price should the company charge for the shredders to maximize revenue? What is the maximum revenue?

Example 4 - maximizing Revenue

Solution
Since

$$\text{revenue} \;=\; \text{price} \;\times\; \text{demand},$$

the mathematical model is built as

$$\text{to maximize } R(x) = p(x) \cdot x = x\left(300 - \frac{x}{30}\right), \text{ subject to .}$$

$$x \geq 0 \text{ and } 300 - \frac{x}{30} \geq 0 \Longrightarrow 0 \leq x \leq 9000$$

$R'(x) = 300 - x/15 \Rightarrow$ critical points: $x = 4500$. It can be checked that $R''(x) = -1/15 < 0$, so $P(4500) = 150$ is the absolute maximum (the only critical point).

Example 5 - Maximizing Profits

Example

An office supply company sells x permanent markers per year at P per marker. The price- demand equation for these markers is given by

$$P = 10 - 0.001x;$$

the total annual cost C of manufacturing x this type of markers is given by

$$C = 5000 + 2x.$$

What price should the company charge to maximize the profit? What is the maximum profit F?

Example 5 - Maximizing Profits

Solution
Since

$$\text{profit} \;=\; \text{revenue} \;-\; \text{cost},$$

The revenue is given by

$$R(x) = P \cdot x = (10 - 0.001x)x = 10x - 0.001x^2.$$

$$\text{to maximize } F(x) = R(x) - C(x) = -5000 + 8x - 0.001x^2,$$

subject to $x \geq 0$ and $p = 10 - 0.001x \geq 0$, that is, $0 \leq x \leq 10000$. It can be calculated that

$$F'(x) = 8 - 0.002x \Rightarrow \text{ critical points:} x = 4000.$$

We can check $F''(x) = -0.002 < 0$, $F(4000) = \$11000$ is the maximum profit and the production then is 4000, and the price is $p = 10 - 0.001 * 4000 = 6$.

Example 6 - Inventory Control

Example

A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs

Example 6 - Inventory Control

Solution - Part 1

This type of problem is called an inventory control problem. One key assumption is that the demand is uniform. For example, if there are 250 working days a year, then the daily demand is $20000/250 = 80$ DVDs. Then there may be two extreme ways to produce this 20,000 copies as follows

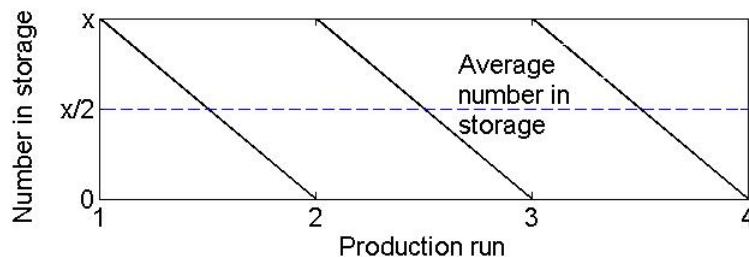|	Produce 20000 once	Produce 80 daily
low set-up cost	low	high
storage cost	high	low

So we can assume

- x is the number of DVDs produced each production run
- y is the number of the production runs

Example 6 - Inventory Control

Solution - Part 2

If it costs \$0.5 to store a DVD for one year, then the total cost to store all DVDs is \$0.25x. Hence

$$C = \text{total cost} = \text{set-up cost} + \text{storage cost} = 200y + 0.25x.$$

The total number of DVD manufactured in one year is $xy = 20000$.

Example 6 - Inventory Control

Solution - Part 3

The mathematical model then can be built up as

to minimize $C = \dfrac{4,000,000}{x} + 0.25x$, subject to $0 < x \leq 20,000$

It can be calculated that

$$C'(x) = 0.25 - \frac{4,000,000}{x^2} \Rightarrow \text{critical points: } x = 4,000.$$

We check by $C''(x) = 8,000,000/x^3 > 0$, so $C(4000)$ is the local minimum and $y = 20,000/4,000 = 5$.
Hence the company can produce 4000 DVDs for 5 times to minimize the cost.

Mini-lecture Question

Erin makes gift boxes from cardboard pieces that measure 8 inches by 10 inches. Equal size squares are cut from each corner of the cardboard piece so that the sides can be folded up to form a rectangular box. What size squares should be cut from each corner of the cardboard piece to maximize the volume of the resulting box? (Hint 1.47 inches)

Mini-lecture Question

A small-town hardware store has a uniform annual demand for 2000 bags of a certain fertilizer. The owner must pay $15 per bag to store the fertilizer at a local facility for one year and a flat fee of $200 to place an order. How many times during the year should the hardware store order fertilizer in order to minimize the total storage reordering costs? (9 times)