MATH 1003 Calculus and Linear Algebra (Lecture 25)

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Antiderivatives

Definition

A function F is an antiderivative of a function f if

$$F'(x) = f(x).$$

Example

Find an antiderivative of $f(x) = 2x$.

Solution

Since $(x^2)' = 2x$, x^2 is an antiderivative. In fact, there are many antiderivatives of $2x$, such as $x^2 + 1$ and $x^2 + 2$.

Antiderivatives

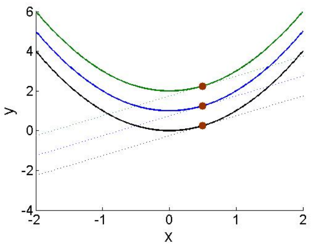
We have the following useful theorem about antiderivatives:

Theorem

If the derivatives of two functions are equal, that is $F'(x) = G'(x)$, then the functions differ by a constant, that is $F(x) = G(x) + C$ for some constant C.

Remark

For any function $f(x)$, if $F(x)$ is its antiderivative, then all antiderivatives must be of the form $F(x) + C$, where C is an arbitrary constant.



Antiderivatives

Example

Let $f(x) = 3x^2$.

(a) Find all antiderivatives of $f(x)$.

(b) Find the antiderivative of $f(x)$ such that the graph of the antiderivative passes through the point (i) $(0, 0)$; (ii) $(0, 1)$; (iii) $(0, 2)$.

Antiderivatives

Solution

(a) Since $(x^3)' = 3x^2$, then the antiderivatives of $f(x)$ are $x^3 + C$, where C is an arbitrary constant.

(b) From (a), $y = x^3 + C$. We consider the following cases:
   (i) If the graph passes through $(0,0)$, then $0 = 0^3 + C$ and hence $C = 0$ i.e. the antiderivative is x^3.
   (ii) If the graph passes through $(0,1)$, $C = 1$ and hence the antiderivative is $x^3 + 1$.
   (iii) If the graph passes through $(0,2)$, $C = 2$ and hence the antiderivative is $x^3 + 2$.

Indefinite Integrals

We use the symbol

$$\int f(x)dx,$$

called the indefinite integral, to represent the family of all antiderivatives of $f(x)$ and write

$$\int f(x)dx = F(x) + C \quad \text{if} \quad F'(x) = f(x).$$

The function $f(x)$ is called the integrand. The symbol dx indicate that the antiderivative is performed with respect to the variable x. The arbitrary constant C is called the constant of integration.

Integration and Differentiation

Notice that indefinite integration and differentiation are inverse operations, except for the addition of the constant of integration. Therefore, when two actions are performed successively, we have

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

$$\int F'(x)dx = F(x) + C.$$

Indefinite Integrals of Basic Functions

Indefinite integrals of some basic functions:

(a) $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$.

(b) $\displaystyle\int e^x dx = e^x + C$

(c) $\displaystyle\int \frac{1}{x} dx = \ln|x| + C$, where $x \neq 0$.

Properties

Basic operations

- $\int kf(x)dx = k\int f(x)dx$, where k is a constant.
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Indefinite Integrals of Basic Functions and Properties

Remarks

- (a)-(c) can be proved by differentiating the right hand side. For example,

$$\left(\frac{x^{n+1}}{n+1}\right)' = (n+1)\cdot\frac{x^n}{n+1} = x^n.$$

- In (c), $\ln|x| = \ln x$ when $x > 0$ and $\ln|x| = \ln(-x)$ when $x < 0$.
- Thanks to the two basic operations, if a function is composed of sum and difference of terms, we can integrate each of these terms separately. For example:

$$\int (3x^3 + x)dx = 3\int x^3dx + \int xdx = \frac{3}{4}x^4 + \frac{1}{2}x^2 + C$$

Some Examples

Example

Find each indefinite integral:

(a) $\int 2dx$

(b) $\int 16e^t dt$

(c) $\int 3x^4dx$

(d) $\int (2x^5 - 3x^2 + 1)dx$

(e) $\int \left(\frac{5}{x} - 4e^x\right) dx$

(f) $\int \left(2x^{\frac{2}{3}} - \frac{3}{x^4}\right) dx$

Some Examples

Solutions Part 1

(a) $\int 2dx = 2\int x^0dx = 2x + C$

(b) $\int 16e^tdt = 16\int e^tdt = 16e^t + C$

(c) $\int 3x^4dx = 3\int x^4dx = \frac{3}{5}x^5 + C$

(d) $\int (2x^5 - 3x^2 + 1)dx = 2\int x^5dx - 3\int x^2dx + \int 1dx$
$= \frac{1}{3}x^6 - x^3 + x + C$

Solutions Part 2

(e) $\displaystyle\int \left(\frac{5}{x} - 4e^x\right) dx = 5\int \frac{1}{x} dx - 4\int e^x dx$

$= 5\ln|x| - 4e^x + C$

(f) $\displaystyle\int \left(2x^{\frac{2}{3}} - \frac{3}{x^4}\right) dx = 2\int x^{\frac{2}{3}} dx - 3\int x^{-4} dx$

$= \displaystyle\frac{6}{5}x^{\frac{5}{3}} + x^{-3} + C$