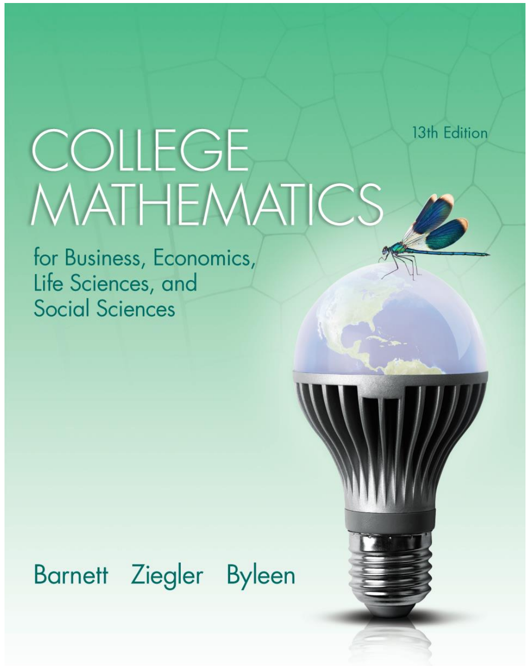


Chapter 13

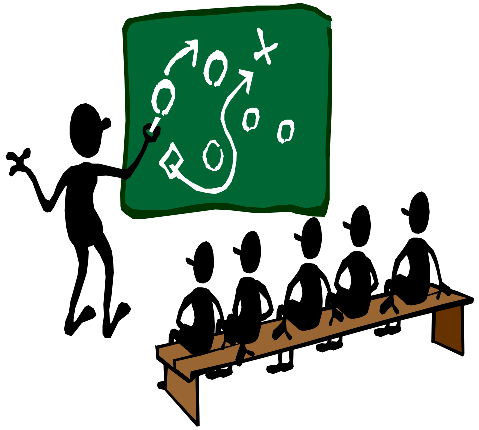
Integration

Section 2

Integration by Substitution

Learning Objectives for Section 5.2 Integration by Substitution

- The student will be able to integrate by reversing the chain rule and by using integration by substitution.
- The student will be able to use additional substitution techniques.
- The student will be able to solve applications.



Reversing the Chain Rule

Recall the chain rule:

$$\frac{d}{dx} f[g(x)] = f'[g(x)]\, g'(x)$$

Reading it backwards, this implies that

$$\int f'[g(x)]\, g'(x)\, dx = f[g(x)] + C$$

Special Cases

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\int [e^{f(x)}] f'(x)\, dx = e^{f(x)} + C$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

Example

$$\int (x^5 - 2)^3 (5x^4)\, dx$$

Note that the derivative of $x^5 - 2$, (i.e. $5x^4$), is present and the integral appears to be in the chain rule form

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq 1$$

with $f(x) = x^5 - 2$ and $n = 3$.

It follows that

$$\int (x^5 - 2)^3 (5x^4)\, dx = \frac{(x^5 - 2)^4}{4} + C$$

Differentials

If $y = f(x)$ is a differentiable function, then

1. The **differential *dx*** of the independent variable x is any arbitrary real number.
2. The **differential *dy*** of the dependent variable y is defined as

$$dy = f'(x)\, dx$$

Examples

1. If $y = f(x) = x^5 - 2$, then

   $dy = f'(x)\, dx = 5x^4\, dx$

2. If $y = f(x) = e^{5x}$, then

   $dy = f'(x)\, dx = 5e^{5x}\, dx$

3. If $y = f(x) = \ln(3x - 5)$, then

   $dy = f'(x)\, dx = \dfrac{3}{3x - 5}\, dx$

Integration by Substitution Example

Example: Find $\int (x^2 + 1)^5 (2x)\, dx$

Example (continued)

Example: Find $(x^2 + 1)^5 (2x)\,dx$

Solution:

For our substitution, let $u = x^2 + 1$,

then $du/dx = 2x$, and $du = 2x\,dx$.

The integral becomes $\int u^5\,du = u^6/6 + C$,

and reverse substitution yields $(x^2 + 1)^6/6 + C$.

General Indefinite Integral Formulas

Very Important!

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^u\,du = e^u + C$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Integration by Substitution

Step 1. Select a substitution that appears to simplify the integrand. In particular, try to select u so that du is a factor of the integrand.

Step 2. Express the integrand entirely in terms of u and du, completely eliminating the original variable.

Step 3. Evaluate the new integral, if possible.

Step 4. Express the antiderivative found in step 3 in terms of the original variable. (Reverse the substitution.)

Example

$\int (x^3 - 5)^4 (3x^2)\,dx$

Step 1 – Select u.

Let $u = x^3 - 5$, then $du = 3x^2\,dx$

Step 2 – Express integral in terms of u.

$\int (x^3 - 5)^4 (3x^2)\,dx = \int u^4\,du$

Step 3 – Integrate.

$\int u^4\,du = u^5/5 + C$

Step 4 – Express the answer in terms of x.

$u^5/5 + C = (x^3 - 5)^5/5 + C$

Example

$\int (x^2 + 5)^{1/2} (2x)\, dx$

Step 1 – Select u.

Let $u = x^2 + 5$, then $du = 2x\, dx$

Step 2 – Express integral in terms of u.

$\int (x^2 + 5)^{1/2} (2x)\, dx = \int u^{1/2}\, du$

Step 3 – Integrate.

$\int u^{1/2}\, du = (2/3)u^{3/2} + C$

Step 4 – Express the answer in terms of x.

$(2/3)u^{3/2} + C = (2/3)(x^2 + 5)^{3/2} + C$

Example

$\int (x^3 - 5)^4 x^2\, dx$

Let $u = x^3 - 5$, then $du = 3x^2\, dx$

We need a factor of 3 to make this work.

$\int (x^3 - 5)^4 x^2\, dx = \mathbf{(1/3)} \int (x^3 - 5)^4 (\mathbf{3}x^2)\, dx$

$= (1/3) \int u^4\, du$

$= (1/3)\, u^5/5 + C$

$= (x^3 - 5)^5/15 + C$

In this problem we had to insert a factor of 3 in order to get things to work out. Caution – a constant can be adjusted, but a variable cannot.

Example

$$\int x^2 e^{4x^3}\, dx$$

Let $u = 4x^3$, then $du = 12x^2\, dx$

We need a factor of 12 to make this work.

$$\int x^2 e^{4x^3}\, dx = \frac{1}{12} \int 12x^2 e^{4x^3}\, dx$$

$$= \frac{1}{12} \int e^u du = \frac{1}{12} e^u + C = \frac{1}{12} e^{4x^3} + C$$

Example

$$\int \frac{x}{(5 - 2x^2)^5}\, dx$$

Let $u = 5 - 2x^2$, then $du = -4x\, dx$

We need a factor of (-4)

$$\int \frac{x}{(5 - 2x^2)^5}\, dx = -\frac{1}{4} \int \frac{-4x}{(5 - 2x^2)^5}\, dx$$

$$= -\frac{1}{4} \int \frac{1}{u^5}\, du = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) u^{-4} + C = \frac{1}{16\left(5 - 2x^2\right)^4} + C$$

Example

$$\int x(x+6)^8\,dx$$

Let $u = x + 6$, then $du = dx$, and $\int x(x+6)^8\,dx = \int xu^8 du$

We need to get rid of the x, and express it in terms of u:

$x = u - 6$, so

$$\int x(x+6)^8\,dx = \int (u-6)u^8 du$$

$$= \int (u^9 - 6u^8)du = u^{10}/10 - 6u^9/9 + C$$

$$= \frac{(x+6)^{10}}{10} - \frac{2(x+6)^9}{3} + C$$

Application

The marginal price of a supply level of x bottles of baby shampoo per week is given by

$$p'(x) = \frac{300}{(3x+25)^2}$$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of \$1.60 per bottle.

Application (continued)

Solution: We need to find

$$p(x) = \int p'(x)dx = \int \frac{300}{(3x+25)^2}\,dx$$

Let $u = 3x + 25$, so $du = 3\,dx$.

$$100\int \frac{3}{(3x+25)^2}\,dx = 100\int \frac{1}{u^2}\,du = \frac{-100}{u} + C$$

$$p(x) = \frac{-100}{3x+25} + C$$

Applications (continued)

Now we need to find C using the fact that 75 bottles sell for \$1.60 per bottle.

$$p(x) = \frac{-100}{3x+25} + C$$

$$p(75) = 1.60$$

$$1.60 = \frac{-100}{3(75)+25} + C$$

We get $C = 2$, so

$$p(x) = \frac{-100}{3x+25} + 2$$