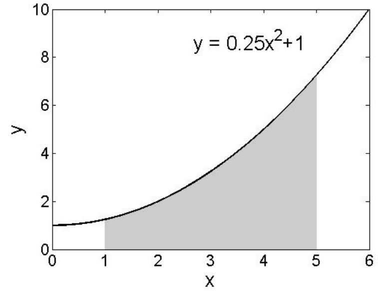
MATH 1003 Calculus and Linear Algebra (Lecture 27)

Maosheng Xiong
Department of Mathematics, HKUST

Area under a graph

Question

Given the graph of $y = f(x) = 0.25x^2 + 1$, as shown in the figure on the right hand side. How can we find the area S under the graph for $1 \leq x \leq 5$?



Left and Right Sum of Rectangles

We can divide the interval $[1, 5]$ by four sub-intervals and compute of the area of the resulting rectangles as follows in two ways.

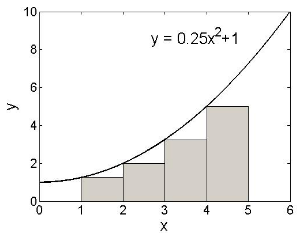


Figure: Left sum of 4 rectangles

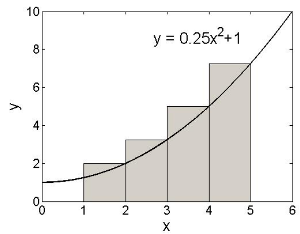


Figure: Right sum of 4 rectangles

- Left sum: $L_4 = f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1$
  $$= 1.25 + 2.00 + 3.25 + 5 = 11.5$$
- Right sum: $R_4 = f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1$
  $$= 2.00 + 3.25 + 5 + 7.25 = 17.5$$

Left and Right Sum of More Rectangles

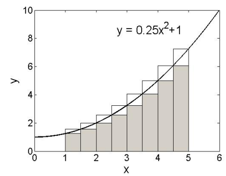


Figure: Sum of 8 rectangles

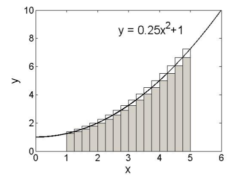


Figure: Sum of 16 rectangles

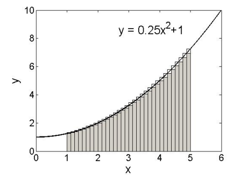


Figure: Sum of 32 rectangles

n	4	8	16	32	100	1000
L_n	11.5	12.88	13.19	13.96	14.21	14.32
R_n	17.5	15.88	15.09	14.71	14.45	14.34

Observation

- $L_4 < L_8 < \cdots <$Area$< \cdots < R_8 < R_4$;
- $|R_n - L_n| \to 0$ as $n \to +\infty$.

Riemann Sum

Idea: Approximate the area S by the sum of areas of rectangles!

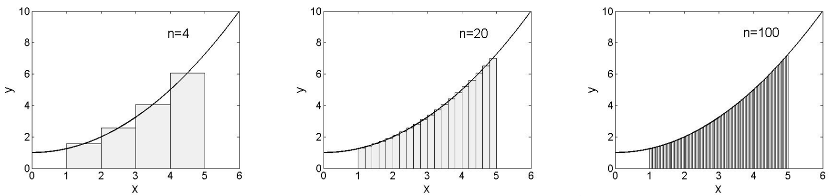
Definition

Suppose $f(x)$ is defined for $a \le x \le b$. One may divide the interval $[a, b]$ into n sub-intervals $[a + (k-1)\Delta x, a + k\Delta x]$, $k = 1, \cdots, n$, where $\Delta x = (b-a)/n$ is the width of each sub-interval. The Riemann sum is defined as

$$S_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x,$$

where $c_k \in [a + (k-1)\Delta x, a + k\Delta x]$.

Examples of Riemann sum of n rectangles:



Definite Integral

Theorem

Let $f(x)$ be a continuous function on [a,b]. As $n \to \infty$, the limit I of Riemann sums S_n for f on $[a, b]$ exists ($I = \lim_{n\to\infty} S_n$).

Definition

The definite integral of f from a to b, denoted by

$$\int_a^b f(x)\, \mathrm{d}x,$$

is defined to be this limit $I = \lim_{n\to\infty} S_n$ of the Riemann sums for f on $[a, b]$. Here $f(x)$ is called the integrand; the lower limit of integration is a; the upper limit of integration is b.

Definite Integral

Definition

Let $f(x)$ be a continuous function on [a,b]. The limit I of Riemann sums for f on $[a, b]$ exists ($I = \lim_{n\to\infty} S_n$). The definite integral of f from a to b is denoted as

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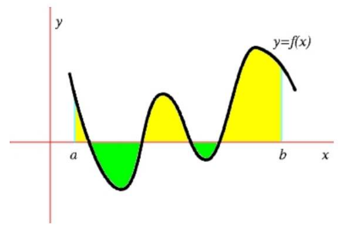
Remarks

- An indefinite integral is $F(x) + C$, a family of functions (antiderivative). However, a definite integral is a real number (the signed area).
- They are intimately related by the Fundamental Theorem of Calculus

Signed Area

$\int_a^b f(x)\, \mathrm{d}x$ represents the cumulative sum of the signed areas between the graph of f and the x-axis form $x = a$ to $x = b$.



The green area below the x-axis will be counted as "negative area" in the sense of the limit of Riemann sum. Hence, we have

$$\int_a^b f(x)\, \mathrm{d}x = \text{signed area} = \text{Yellow area} - \text{Green area}$$

Properties of Definite Integrals

Properties

1. $\int_a^a f(x)\,\mathrm{d}x = 0$
2. $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

Example

Given $\int_0^2 x^2\,\mathrm{d}x = 8/3$, $\int_2^3 x^2\,\mathrm{d}x = 19/3$, then

$$\int_0^3 x^2\,\mathrm{d}x = \int_0^2 x^2\,\mathrm{d}x + \int_2^3 x^2\,\mathrm{d}x = \frac{8}{3} + \frac{19}{3} = 9.$$

Homework

- Web of work, Problem 7 (deadline: 24-Nov-2017)
- Web of work, Problem 8 (deadline: 30-Nov-2017)
- More exercises may be found in Ch 13.1-2 of the textbook.