MATH 1003 Calculus and Linear Algebra (Lecture 28)

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Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Suppose $\int f(x)\,\mathrm{d}x = F(x) + C$. *Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$$

Remarks

- Thanks to the Fundamental theorem, a definite integral can be evaluated by the corresponding indefinite integral, without the need of going through the difficult process of taking the limit of its Riemann sum.
- Area of an irregular shape can be found by computing a suitable definite integral.

Properties of Definite Integrals

(a) $\int_a^a f(x)\,\mathrm{d}x = 0$

(b) $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

(c) $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$

(d) $\int_a^b [f(x) \pm g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$

(e) $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

Warm-up Examples

Example

Evaluate each of the following definite integrals:

(a) $\int_0^2 \sqrt{x}\,\mathrm{d}x$

(b) $\int_{-1}^{\frac{1}{2}} 3e^x\,\mathrm{d}x$

(c) $\int_1^4 \frac{2-x^2}{x}\,\mathrm{d}x$

Warm-up Examples

Solutions

(a) $\int \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$,

so $\int_0^2 \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^2 = \frac{2}{3}2^{\frac{3}{2}}$.

(b) $\int 3e^x\,dx = 3e^x + C$,

so $\int_{-1}^{\frac{1}{2}} 3e^x\,dx = 3e^x\Big|_{-1}^{\frac{1}{2}} = 3(e^{\frac{1}{2}} - e^{-1})$.

Warm-up Examples

Solutions

(c) $\int \frac{2-x^2}{x}\,dx = \int \left(\frac{2}{x} - x\right)\,dx = 2\int \frac{1}{x}\,dx - \int x\,dx$

$= 2\ln|x| - \frac{x^2}{2} + C$,

so $\int_1^4 \frac{2-x^2}{x}\,dx = \left(2\ln|x| - \frac{x^2}{2}\right)\Big|_1^4 =$

$(2\ln 4 - \frac{4^2}{2}) - (2\ln 1 - \frac{1^2}{2}) = 2\ln 4 - \frac{15}{2}$.

Definite integrals and Substitution

Example

Evaluate each of the following definite integrals

(a) $\int_0^1 \frac{x}{x^2+2}\,dx$

(b) $\int_1^2 xe^{2x^2-3}\,dx$

(c) $\int_{-2}^0 x\sqrt{2x+4}\,dx$

Definite integrals and Substitution

Method 1 for (a)

Step 1, evaluate the indefinite integral:

Let $u = x^2 + 2$. We get $du = 2x\,dx$. Hence $x\,dx = \frac{1}{2}\,du$. Then

$$\int \frac{x}{x^2+2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+2) + C.$$

Step 2, evaluate the definite integral:

$$\int_0^1 \frac{x}{x^2+2}\,dx = \frac{1}{2}\ln(x^2+2)\Big|_0^1 = \frac{\ln 3 - \ln 2}{2}.$$

Method 2 for (a)

Let $u = x^2 + 2$. We get $\mathrm{d}u = 2x\,\mathrm{d}x$. Hence $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. We also change the limits of the integration as follows:

- $x = 0$ implies that $u = 0^2 + 2 = 2$,
- $x = 1$ implies that $u = 1^2 + 2 = 3$.

Then

$$\int_0^1 \frac{x}{x^2+2}\,\mathrm{d}x = \frac{1}{2}\int_2^3 \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\left(\ln|u|\right)\Big|_2^3 = \frac{1}{2}\left(\ln 3 - \ln 2\right).$$

Method 1 for (b)

Step 1, evaluate the indefinite integral:

Let $u = 2x^2 - 3$. We get $\mathrm{d}u = 4x\,\mathrm{d}x$. Hence $x\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$. Then

$$\int xe^{2x^2-3}\,\mathrm{d}x = \frac{1}{4}\int e^u\,\mathrm{d}u = \frac{1}{4}e^u + C = \frac{1}{4}e^{2x^2-3} + C.$$

Step 2, evaluate the definite integral:

$$\int_1^2 xe^{2x^2-3}\,\mathrm{d}x = \frac{1}{4}e^{2x^2-3}\Big|_1^2 = \frac{e^5 - e^{-1}}{4}.$$

Method 2 for (b)

Let $u = 2x^2 - 3$. We get $\mathrm{d}u = 4x\,\mathrm{d}x$. Hence $x\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$. We also change the limits of the integration as follows:

- $x = 1$ implies that $u = 2 * 1^2 - 3 = -1$,
- $x = 2$ implies that $u = 2 * 2^2 - 3 = 5$.

$$\int_1^2 xe^{2x^2-3}\,\mathrm{d}x = \frac{1}{4}\int_{-1}^5 e^u\,\mathrm{d}u = \left(\frac{1}{4}e^u\right)\Big|_{-1}^5 = \frac{1}{4}\left(e^5 - e^{-1}\right).$$

Method 1 for (c)

Step 1, evaluate the indefinite integral:
Let $u = 2x + 4$ so that $x = (u-4)/2$. We get $\mathrm{d}u = 2\,\mathrm{d}x$. Hence $\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. Then

$$\int x\sqrt{2x+4}\,\mathrm{d}x = \frac{1}{2}\int \frac{u-4}{2}\cdot\sqrt{u}\,\mathrm{d}u = \frac{1}{4}\int u^{3/2}\,\mathrm{d}u - \int u^{1/2}\,\mathrm{d}u$$

$$= \frac{1}{4}\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{u^{5/2}}{10} - \frac{2u^{3/2}}{3} + C$$

$$= \frac{(2x+4)^{5/2}}{10} - \frac{2(2x+4)^{3/2}}{3} + C.$$

Definite integrals and Substitution

Method 1 for (c)

Step 2, evaluate the definite integral:

$$\int_{-2}^{0} x\sqrt{2x+4}\,\mathrm{d}x = \left(\frac{(2x+4)^{5/2}}{10} - \frac{2(2x+4)^{3/2}}{3}\right)\Bigg|_{-2}^{0} = \left(\frac{(4)^{5/2}}{10} - \frac{2(4)^{3/2}}{3}\right) - 0 = -\frac{32}{15}.$$

Definite integrals and Substitution

Method 2 for (c)

Step 1, evaluate the indefinite integral:
Let $u = 2x + 4$ so that $x = (u-4)/2$. We get $\mathrm{d}u = 2\,\mathrm{d}x$. Hence $\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. We also change the limits of the integration as follows:

- $x = -2$ implies that $u = 2*(-2) + 4 = 0$,
- $x = 0$ implies that $u = 2*0 + 4 = 4$.

Then

$$\int_{-2}^{0} x\sqrt{2x+4}\,\mathrm{d}x = \frac{1}{2}\int_{0}^{4} \frac{u-4}{2}\cdot\sqrt{u}\,\mathrm{d}u = \frac{1}{4}\int_{0}^{4} u^{3/2}\,\mathrm{d}u - \int_{0}^{4} u^{1/2}\,\mathrm{d}u$$

$$= \left(\frac{1}{4}\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\Bigg|_{0}^{4} = \left(\frac{u^{5/2}}{10} - \frac{2u^{3/2}}{3}\right)\Bigg|_{0}^{4}$$

$$= \frac{(4)^{5/2}}{10} - \frac{2(4)^{3/2}}{3} - 0 = -\frac{32}{15}.$$

Problems for thought

Example

(a) $\displaystyle\int_{1}^{2} \frac{x+1}{2x^2+4x+4}\,\mathrm{d}x$

(b) $\displaystyle\int_{e}^{e^2} \frac{(\ln t)^2}{t}dt$

(Answer: (a) $(\ln 2)/4$, (b) $7/3$)

Problems for thought

(a)

Let $u = 2x^2 + 4x + 4$ so that $\mathrm{d}u = 4(x+1)\,\mathrm{d}x$. We get $(x+1)\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$. We also change the limits of the integration as follows:

- $x = 1$ implies that $u = 2*1^2 + 4*1 + 4 = 10$,
- $x = 2$ implies that $u = 2*2^2 + 4*2 + 4 = 20$.

Then

$$\int_{1}^{2} \frac{x+1}{2x^2+4x+4}\,\mathrm{d}x = \frac{1}{4}\int_{10}^{20} \frac{1}{u}\,\mathrm{d}u = \frac{1}{4}(\ln u)\Big|_{10}^{20} = \frac{1}{4}(\ln 20 - \ln 10) = \frac{\ln 2}{4}.$$

Problems for thought

(b)

Let $u = \ln t$ so that $\mathrm{d}u = \frac{1}{t}dt$. We also change the limits of the integration as follows:

- $t = e$ implies that $u = \ln e = 1$,
- $t = e^2$ implies that $u = \ln e^2 = 2$.

Then

$$\begin{aligned} \int_e^{e^2} \frac{(\ln t)^2}{t} dt &= \int_1^2 u^2 \, \mathrm{d}u = \frac{1}{3}u^3 \Big|_1^2 \\ &= \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}. \end{aligned}$$