MATH 1003 Calculus and Linear Algebra (Lecture 29)

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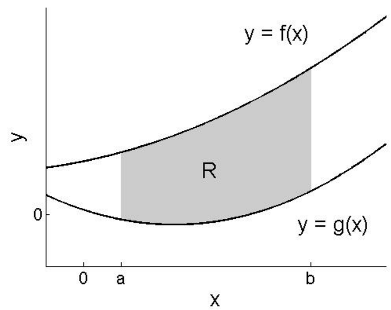
Area between two curves

Consider the area R bounded by $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, for $a \leq x \leq b$:

$$R = (\text{area under } f(x)) - (\text{area under } g(x))$$
$$= \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$



A typical example

Question

Find the area R bounded by $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.

Idea

It is important to find the sign chart of $f(x) - g(x)$, that is, it is important to know for which x one has $f(x) - g(x) \geq 0$ and for which x one has $f(x) - g(x) \leq 0$ where $a \leq x \leq b$. For example, if we know that

$$\begin{aligned} f(x) - g(x) \geq 0 \quad & \text{if } a \leq x \leq c, \\ f(x) - g(x) \leq 0 \quad & \text{if } c \leq x \leq b, \end{aligned}$$

Then the area R is

$$R = \int_a^c f(x) - g(x)\, dx + \int_c^b g(x) - g(x)\, dx.$$

Examples - Areas between a curve and x-axis

Example

Find the area bounded by $y = 6x - x^2$ and $y = 0$ for $1 \leq x \leq 4$.

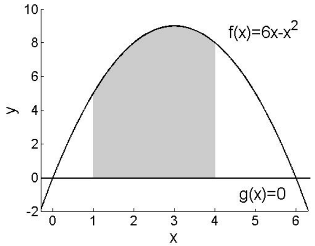
Remark

Let $f(x) = 6x - x^2$, $g(x) = 0$. It is important to find the sign chart of $f(x) - g(x) = (6x - x^2) - 0$, that is, it is important to know for which x one has $f(x) - g(x) > 0$ and for which x one has $f(x) - g(x) < 0$ where $1 \leq x \leq 4$.

Examples - Areas between a curve and x-axis

Solution

We now target the two curves. One is characterised by $y = 6x - x^2$, the other by $y = 0$ (x-axis). We need to decide which $f(x)$ and $g(x)$ are, respectively.



As $1 \le x \le 4$, $6x - x^2 \ge 0$. So let

$$f(x) = 6x - x^2, \text{ and } g(x) = 0.$$

Therefore, the area is

$$\int_1^4 (6x - x^2)dx = \left(3x^2 - \frac{x^3}{3}\right)\Big|_1^4 = 24.$$

Examples - Areas between a curve and x-axis

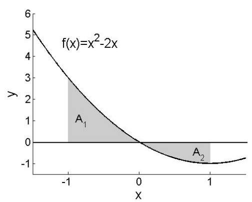
Example

Find the area between the graph of $y = x^2 - 2x$ and the x-axis over the interval $[-1, 1]$

Examples - Areas between a curve and x-axis

Solution - Part 1

If we choose $f(x) = x^2 - 2x$, then we first check $x^2 - 2x = 0$ and we have two zeros $x = 2$ or $x = 0$. In this case,



$$f(x) \begin{cases} \ge 0, & 0 \le x \le 1; \\ \le 0, & -1 \le x \le 0. \end{cases}$$

Therefore, the total area is the sum of two parts:

$$A_1 = \int_{-1}^0 f(x) - 0dx$$

and

$$A_2 = \int_0^1 0 - f(x)dx.$$

Examples - Areas between a curve and x-axis

Solution - Part 2

Since

$$\int (x^2 - 2x)dx = \frac{x^3}{3} - x^2 + C,$$

the required area is

$$\begin{aligned} &\int_{-1}^0 (x^2 - 2x)dx - \int_0^1 (x^2 - 2x)dx \\ &= \left(\frac{x^3}{3} - x^2\right)\Big|_{-1}^0 + \left(x^2 - \frac{x^3}{3}\right)\Big|_0^1 \\ &= -\left(\frac{(-1)^3}{3} - (-1)^2\right) + \left(1^2 - \frac{1^3}{3}\right) = 2. \end{aligned}$$

Examples - Areas bounded by two curves

Example

Find the area bounded by the graphs of $y = 5 - x^2$ and $y = 2 - 2x$.

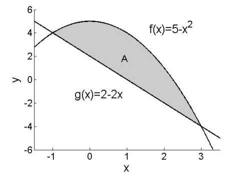
Examples - Areas bounded by two curves

Solution - Part 1

Similar to previous examples, we need to solve
$(5 - x^2) - (2 - 2x) = 0 \Rightarrow x = -1$ and 3.

Further we find

$$(5-x^2)-(2-2x) \begin{cases} \geq 0, & -1 \leq x \leq 3 \\ \leq 0, & x \geq 3 \text{ or } x \leq -1. \end{cases}$$



So the bounded region is $-1 \leq x \leq 3$, with $f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.

Examples - Areas bounded by two curves

Solution - Part 2

Since

$$\int [(5 - x^2) - (2 - 2x)]dx = 3x + x^2 - \frac{x^3}{3}$$

we have

$$A = \int_{-1}^{3} [(5 - x^2) - (2 - 2x)]dx = \left(3x + x^2 - \frac{x^3}{3}\right)\Big|_{-1}^{3} = \frac{32}{3}.$$

Examples - Areas bounded by two curves

Example

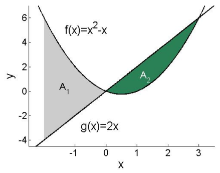
Find the area bounded by the graphs of $y = x^2 - x$ and $y = 2x$ for $-2 \leq x \leq 3$.

Examples - Areas bounded by two curves

Solution - Part 1

If we set $f(x) = x^2 - x$ and $g(x) = 2x$, then solving $f(x) - g(x) = x^2 - 3x = 0$ gives rise to $x = 0$ or $x = 3$.

Further we find here



$$f(x) - g(x) \begin{cases} \geq 0, & -2 \leq x \leq 0 \\ \leq 0, & 0 \leq x \leq 3. \end{cases}$$

The total area is computed separately:

$$A_1 = \int_{-2}^{0} (f(x) - g(x))dx$$

and

$$A_2 = \int_{0}^{3} (g(x) - f(x))dx.$$

Examples - Areas bounded by two curves

Solution - Part 2

Since

$$\int (x^2 - 3x)dx = \frac{x^3}{3} - \frac{3x}{2},$$

we have

$$\int_{-2}^{0} (x^2 - 3x)dx - \int_{0}^{3} (x^2 - 3x)dx$$
$$= (\frac{x^3}{3} - \frac{3x^2}{2})\Big|_{-2}^{0} + (\frac{3x^2}{2} - \frac{x^3}{3})\Big|_{0}^{3} = \frac{79}{6}.$$