MATH 1003 Calculus and Linear Algebra (Lecture 30)

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1. Average Value of a Continuous Function

Definition

Let $f(x)$ be a continuous function on $[a, b]$. The average value of $f(x)$ on $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Interpretation

One may divide the interval $[a, b]$ into n sub-intervals. Then from each subinterval I_k, we sample one point x_k and take the average:

$$\bar{A}_n = \frac{f(x_1)+\cdots+f(x_n)}{n} = \underbrace{(f(x_1)+\cdots+f(x_n))\cdot\frac{b-a}{n}}_{\text{Riemman sum}}\cdot\frac{1}{b-a}.$$

Therefore, $\lim_{n\to\infty}\bar{A}_n = \frac{1}{b-a}\int_a^b f(x)dx$

Average Value of a Continuous Function

Example

Given the demand function

$$p = D(x) = 100e^{-0.05x},$$

find the average price (in dollars) over the demand interval $[40, 60]$.

Solution

The average price is

$$\begin{aligned}\bar{p} &= \frac{1}{b-a}\int_a^b D(x)dx = \frac{1}{60-40}\int_{40}^{60} 100e^{-0.05x}dx \\ &= \frac{100}{20}\left(\frac{e^{-0.05x}}{-0.05}\right)\Bigg|_{40}^{60} = 100\left(e^{-2}-e^{-3}\right) \approx \$8.55.\end{aligned}$$

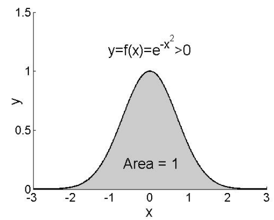
2. Probability Density Functions

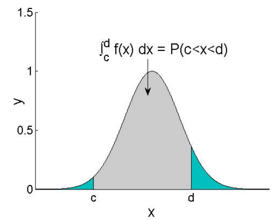
A probability density function must satisfy the following three conditions:

1. $f(x) \geq 0$ for all real x;
2. The area under the graph of $f(x)$ over the interval $(-\infty, \infty)$ is exactly 1;
3. If $[c, d]$ is a sub-interval of $(-\infty, \infty)$, then the probability of x falling in $[c, d]$ is defined by:

$$P(c \leq x \leq d) = \int_c^d f(x)dx$$

Example





A Mini-Example

Example

Suppose there are two stocks A and B for investment. Historical data have shown the following information.

- Stock A is now \$60 per share, and in one year its share price increase x satisfies a probability density distribution

$$f_A(x) = \begin{cases} \frac{3}{4000}\left(75 + 10x - x^2\right) & : \quad -5 \le x \le 15, \\ 0 & : \quad \text{for } x \text{ elsewhere.} \end{cases}$$

- Stock B is now \$70 per share, and in one year its share price increase x satisfies a probability density distribution:

$$f_B(x) = \begin{cases} \frac{3}{500}\left(9 + 8x - x^2\right) & : \quad -1 \le x \le 9 \\ 0 & : \quad \text{for } x \text{ elsewhere.} \end{cases}$$

(a) Find the probabilities of a loss investing in A and B in one year.

A Mini-Example

Example

(a) Find the probabilities of a loss investing in A and B in one year.

(b) Find the probabilities of a 10% gain investing in each of the stocks A and B.

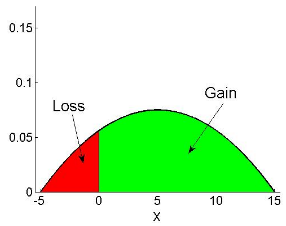
A Mini-Example

Solution for (a)

For A it can be checked that $f_A(x) \ge 0$ and $\int_{-5}^{15} f_A(x)dx = 1$. Investment loss means $x \le 0$.

$$P_A(x \le 0) = \int_{-5}^{0} \frac{3(75 + 10x - x^2)}{4000} dx$$

$$= \frac{3}{4000}\left(75x + 5x^2 - \frac{x^3}{3}\right)\Big|_{-5}^{0} = 0.156$$



Interpretation The chance of loss investing in A in one year is 15.6%.

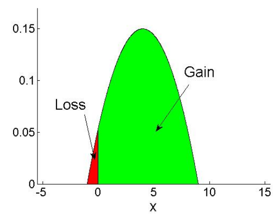
A Mini-Example

Solution for (a)

For B it can be checked that $f_B(x) \ge 0$ and $\int_{-1}^{9} f_B(x)dx = 1$. Investment loss means $x \le 0$.

$$P_B(x \le 0) = \int_{-1}^{0} \frac{3(9 + 8x - x^2)}{500} dx$$

$$= \frac{3}{500}\left(9x + 4x^2 - \frac{x^3}{3}\right)\Big|_{-1}^{0} = 0.0280$$



Interpretation The chance of loss investing in B in one year is 2.8%.

Which one is more risky?

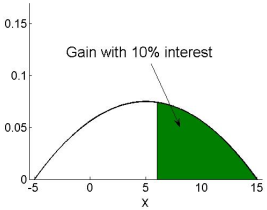
A Mini-Example

Solution for (b)

10% profit from A means $x \geq 60 \times 0.1 = 6$.

$$P_A(x \geq 6) = \int_6^{15} \frac{3(75 + 10x - x^2)}{4000} dx$$

$$= \frac{3}{4000}\left(75x + 5x^2 - \frac{x^3}{3}\right)\Bigg|_6^{15} = 0.4253$$



Interpretation The chance of 10% gain investing in A in one year is 42.53%.

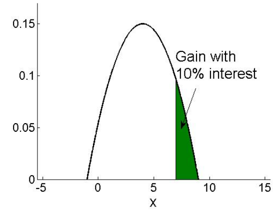
A Mini-Example

Solution for (b)

10% profit from B means $x \geq 70 \times 0.1 = 7$.

$$P_B(x \geq 7) = \int_7^{9} \frac{3(9 + 8x - x^2)}{500} dx$$

$$= \frac{3}{500}\left(9x + 4x^2 - \frac{x^3}{3}\right)\Bigg|_7^{9} = 0.1040$$



Interpretation The chance of 10% gain investing in B in one year is 10.4%.

Combination of A and B - Portfolios

Example

Example

The life expectancy (in years) of a microwave oven is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{2}{(x+2)^2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that a randomly selected microwave oven lasts at most 6 years.

(b) Find the probability that a randomly selected microwave oven lasts 6 to 12 years.

Example

Solution (a) Let Y be the life expectancy of a microwave oven. Then

$$\begin{aligned} \Pr(Y \leq 6) &= \int_0^6 \frac{2}{(x+2)^2} dx \\ &= \int_0^6 2(x+2)^{-2} dx \\ &= \frac{-2}{x+2}\Bigg|_0^6 \\ &= -\frac{2}{8} + 1 = 0.75. \end{aligned}$$

Example

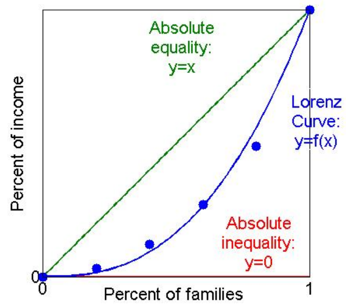
Solution (b)

$$
\begin{aligned}
\Pr(6 \le Y \le 12) &= \int_6^1 2\frac{2}{(x+2)^2}\,dx \\
&= \left.\frac{-2}{x+2}\right|_6^{12} \\
&= -\frac{2}{14} + \frac{1}{4} = 0.107.
\end{aligned}
$$

Income Distribution of a Society

The following is the family income distribution in U.S., 2006

Income Level	x	y
$< \$20,000$	0.20	0.03
$< \$38,000$	0.40	0.12
$< \$60,000$	0.60	0.27
$< \$97,000$	0.80	0.49



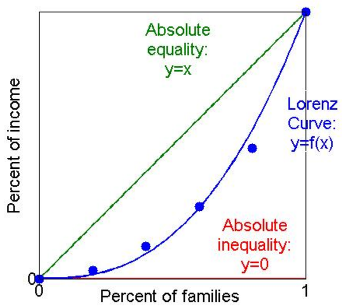
The variable x represents the cumulative percentage of families at or below a given income level, and y represents the cumulative percentage of total family income received. For example, the point $(0.40, 0.12)$ means that the bottom 40% of families (incomes under $ 38,000) received 12% of the total income for all families.

Income Distribution of a Society

The blue curve in the graph of $y = f(x)$ is called the Lorenz curve. For example, it may be expressed by

$$f(x) = x^{\alpha}$$

with $\alpha \ge 1$.



The variable x represents the cumulative percentage of families at or below a given income level, and y represents the cumulative percentage of total family income received. For example, $(0.40, 0.12)$ is the point on the Lorenz curve and means that the bottom 40% of families received 12% of the total income for all families.

Absolute Equality and Inequality

Remarks

- Any Lorenz curve is below the 45 degree line.
- If the income were distributed with absolute equality, the Lorenz curve would coincide with the 45 degree line.
- If the income were distributed with absolute inequality, the Lorenz curve would coincide with the horizontal axis and the right vertical axis.

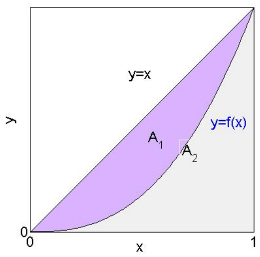
Gini Index

Let us define the Gini index, a measurement of the degree of inequality in the distribution of income in a society:

Definition

If the Lorenz curve is given by $y = f(x)$, then Gini index (G) is defined to be

$$G = 2\int_0^1 (x - f(x))dx$$



Remarks

- G is the ratio between A_1 (the area enclosed by $y = x$ and $y = f(x)$) and A_2 (the area enclosed by $y = x$, $y = 0$ and $x = 1$) and $0 \le G \le 1$.
- As G increases, the degree of inequality in the distribution of income increases.

Gini Index

Example

A country is planning changes in tax structure in order to provide a more equitable distribution of income. The two Lorenz curves are: $f(x) = x^{2.3}$ currently and $g(x) = 0.4x + 0.6x^2$ proposed. Will the proposed changes work?

Gini Index

Solution

Currently, the Gini index is

$$2\int_0^1 (x - x^{2.3})dx = 2\left(\frac{x^2}{2} - \frac{x^{3.3}}{3.3}\right)\Big|_0^1 = 0.39$$

After the proposed changes, the Gini index is

$$2\int_0^1 [x - (0.4x + 0.6x^2)]dx = 2\left(\frac{0.6x^2}{2} - \frac{0.6x^3}{3}\right)\Big|_0^1 = 0.20$$

Therefore, the proposed changes will work because the Gini index becomes lower.

Consumers' Surplus

Example

If $(\bar{x}, \bar{p})$ is a point on the graph of the price-demand equation $p = D(x)$, the consumers' surplus CS at a price level of $\bar{p}$ is

$$CS(\bar{x}) = \int_0^{\bar{x}} D(x) - \bar{p}\, dx.$$

This is the area between $p = \bar{p}$ and $p = D(x)$ from $x = 0$ to $x = \bar{x}$. The consumers' surplus represents the total savings to consumers who are willing to pay more than $\bar{p}$ for the product but are still able to buy the product for $\bar{p}$.

Example

Find the consumers' surplus at a price level of $\bar{p} = 120$ for the price-demand equation

$$p = D(x) = 200 - 0.02x.$$

Solution First, find the demand when the price is $\bar{p} = 120$:

$$120 = \bar{p} = 200 - 0.02\bar{x} \Longrightarrow \bar{x} = 4000.$$

Solution Second, find the consumers' surplus:

$$\begin{aligned} CS(\bar{x}) &= \int_0^{\bar{x}} D(x) - \bar{x}\, dx \\ &= \int_0^{4000} 200 - 0.02x - 120\, dx \\ &= \int_0^{4000} 80 - 0.02x\, dx \\ &= (80x - 0.01x^2)\Big|_0^{4000} \\ &= 320000 - 160000 = \$160000. \end{aligned}$$