MATH 1003 Calculus and Linear Algebra (Lecture 31)

Maosheng Xiong
Department of Mathematics, HKUST

Integration by Parts

The product rule of differentiation:

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Integrating on both sides with respect to x:

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

If we set $u = u(x)$ and $v = v(x)$, we have the integration by parts formula:

$$\int u \underbrace{dv}_{v'dx} = uv - \int v \underbrace{du}_{u'dx}$$

Example

Find $\int xe^x dx$.

Integration by Parts - Examples

Solution

0). Observe that other methods do not work, so we are forced to use integration by parts.

1). We split "$xe^x dx$" into two parts: the "u" part and the "dv" part. which is u and which is dv? This depends on experience. We choose

$$u = x, \quad dv = e^x dx.$$

Thus

$$du = dx, \text{ and } v = \int e^x dx = e^x.$$

According to the integration by parts formula, we have

$$\int xe^x dx = uv - \int v\, du = xe^x - \int e^x dx = xe^x - e^x + C$$

What will you get by choosing $u = e^x$ and $dv = xdx$?

Integration by Parts - Examples

Example

Find $\int x \ln x dx$.

Integration by Parts - Examples

Solution - Part 1

We let

$$u = \ln x, \quad dv = x dx.$$

Then we have

$$du = \frac{1}{x} dx, \quad v = \int x dx = \frac{x^2}{2}.$$

Therefore, by the integration by parts formula, we have

$$\int x \ln x dx = uv - \int v\, du = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx.$$

Integration by Parts - Examples

Solution - Part 2

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$\Rightarrow \int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

Remarks

1). This method works for $\int x^p \ln x dx$, where $p \neq -1$.

2). What happens if $p = -1$? That is, how do you compute $\int x^{-1} \ln x dx$? (Hint: substitution: $u = \ln x$)

Example Calculate $\int \ln x dx$

Solution: Let $u = \ln x$ and $dv = dx$, so that $v = x$. You find

$$\int \ln x dx = x \ln x - x + C$$

Repeated Integration by Parts

Example

Find $\int x^2 e^{-2x} dx$.

Repeated Integration by Parts

Solution - Part 1

Let $u = x^2$ and $dv = e^{-2x}dx$. Then we have $du = 2xdx$ and

$$v = \int e^{-2x}dx \overset{t=-2x}{=} \frac{-1}{2}\int e^t dt = -\frac{1}{2}e^{-2x}$$

Using the integration by parts formula, we get

$$\int x^2 e^{-2x}dx = -\frac{x^2e^{-2x}}{2} - \int 2x\frac{-e^{-2x}}{2}dx = -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}dx.$$

Repeated Integration by Parts

Solution - Part 2

The integral on the right hand side is still not simple enough. We need to use integration by parts once more to calculate $\int xe^{-2x}dx$. Let $u = x$ and $v'dx = e^{-2x}dx$. We get

$$du = dx \text{ and } v = -\frac{1}{2}e^{-2x}.$$

Using the integration by parts formula, we get

$$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}.$$

Incorporating the above formula with the last formula in the previous slide gives

$$\int x^2e^{-2x}dx = -\frac{e^{-2x}}{4}\left(1 + 2x + 2x^2\right).$$

Integration by Parts for Definite Integrals

The following is the integration by parts formula for definite integrals:

$$\int_a^b uv'dx = uv\Big|_a^b - \int_a^b vu'dx$$

or

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

Example

Find $\int_1^2 x(\ln x)^2dx$.

Integration by Parts for Definite Integrals

Solution

Let $u = (\ln x)^2$, $dv = xdx$. Then $du = 2\ln x/x\,dx$ and $v = \int xdx = x^2/2$. Therefore, we have

$$\int_1^2 x(\ln x)^2dx = \frac{x^2(\ln x)^2}{2}\Big|_1^2 - \int_1^2 x\ln xdx = 2(\ln 2)^2 - \int_1^2 x\ln xdx.$$

To calculate $\int_1^2 x\ln xdx$, let $u = \ln x$ and $v'dx = xdx$ and we have $du = 1/x\,dx$ and $v = x^2/2$. Hence we obtain

$$\int_1^2 x\ln xdx = \frac{x^2\ln x}{2}\Big|_1^2 - \frac{1}{2}\int xdx = 2\ln 2 - \frac{x^2}{4}\Big|_1^2 = 2\ln 2 - \frac{3}{4}.$$

The final answer is $\int_1^2 x(\ln x)^2dx = 2(\ln 2)^2 - 2\ln 2 + 3/4$.

Summary of the integration by parts technique

Key: properly split the integrand to find u and dv

- To calculate $\int x^n e^{ax} dx$, we need

$$u = x^n, \quad dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax};$$

- To calculate $\int x^p (\ln x)^q dx$, we need

$$u = (\ln x)^q, \quad dv = x^p dx \Rightarrow v = \frac{1}{p+1} x^{p+1}.$$