MATH 1003 Review: Part 3. The Derivatives of Functions

Maosheng Xiong
Department of Mathematics, HKUST

Big Picture I

What would the following questions remind you?

1. Concepts:
   - limit, one-sided limit,
   - continuity,
   - derivative, instantaneous rate of change, slope of tangent line, velocity,
   - e,
   - Continuous compound interest model
   - vertical and horizontal asymptotes
2. Exponential and logarithmic functions: domain and range of $e^x, \ln x$, and derivatives
3. critical point, inflexion point,

Big Picture I

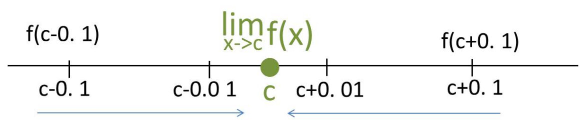
What would the following questions remind you?

1. Techniques of taking derivatives: power rule, product rule, quotient rule, chain rule, implicit differentiation.
2. Rates of change problems.
3. The first and the second derivative test.
4. Optimization problems.
5. Graph sketching: intervals that function is increasing/decreasing, concavity.

Introduction to Derivatives

- $\lim_{x\to c}$ f(x) = L (L.13, Ch10.4)

$c-0.1$	$c-0.01$	$\cdots$	$c+0.01$	$c+0.1$
$f(c-0.1)$	$f(c-0.01)$	$\cdots$	$f(c+0.01)$	$f(c+0.1)$



- Derivative of $y = f(x)$(L.13, Ch10.4) is defined by

$$\underbrace{f'(x)}_{notation} = \underbrace{\frac{dy}{dx}}_{notation} = \underbrace{\frac{df(x)}{dx}}_{notation} = \underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}_{definition}.$$

- Meaning of the derivative of $y = f(x)$:
  - $f'(a)$ - instantaneous rate of change of $f(x)$ at time a (physics)
  - $f'(a)$ - slope of the tangent line to $f(x)$ at $(a, f(a))$ (graphing)

Functions associated with the constant e

- Defining constant e(L.15, Ch11.1):

$$e = \lim_{s \to 0}(1+s)^{1/s} \approx 2.718\cdots.$$

- Exponential functions (L.15, Ch11.2)
  - with base e: $y = e^x$, $(e^x)' = e^x$
  - with base a: $y = a^x$, $(a^x)' = a^x \ln a$ $(a > 0)$
- Logarithmic functions (L.15, Ch11.2)
  - with base e: $y = \ln x$, $(\ln x)' = 1/x$
  - with base a: $y = \log_a x$, $(\log_a x)' = 1/(x \ln a)$ $(a > 0, x > 0)$

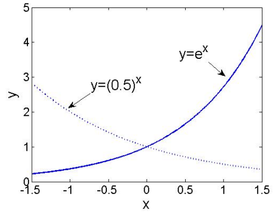


Figure: Exponential functions

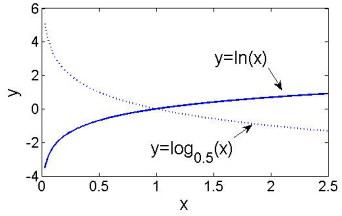


Figure: Logarithmic functions

Searching for the Derivative of a Function (Part 1)

- Basic operation (L.14, Ch10.5):

$$(u(x) + v(x))' = u'(x) + v'(x)$$

$$(k(u(x)))' = ku'(x)$$

- Power rule (L.14, Ch10.5):

$$(x^n)' = nx^{n-1}$$

- Product rule (L.16, Ch11.3):

$$(u(x)v(x))' = u'(x)v(x) + v'(x)u(x)$$

- Quotient rule (L.16, Ch11.3):

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{(v(x))^2}$$

Searching for the Derivative of a Function (Part 2)

The chain rule (L.17, Ch11.4):

- Special cases:
  - general power rule, $y = f(x) = (u(x))^n$:

$$\frac{dy}{dx} = f'(x) = n(u(x))^{n-1} \cdot u'(x);$$

  - exponential type, $y = f(x) = e^{u(x)}$:

$$\frac{dy}{dx} = f'(x) = e^{u(x)} \cdot u'(x);$$

  - logarithmic type, $y = f(x) = \ln(u(x))$:

$$\frac{dy}{dx} = f'(x) = \frac{1}{u(x)} \cdot u'(x).$$

- General formula:

$$(g(u(x)))' = g'(u(x)) \cdot u'(x).$$

Generalisation of the Idea of Derivatives

Implicit Differentiation (L.18, Ch11.5):

$$F(x, y) = \text{constant}$$

- y is an implicit function of x
- Evaluation of dy/dx at $(x, y) = (a, b)$:

$$\text{key step: } \frac{d}{dx}F(x, y) = 0,$$

  where the calculation of the derivative of terms including y needs the chain rule (Ref to procedures introduced in L. 18).

Rate of change (L.19, Ch11.6):

- An independent variable t (normally time)
- A number of inter-related dependent variables x, y, z, $\cdots$.
- Rate of change on one dependent variable x is obtained by taking derivative to $x = F(y, z, \cdots)$ with respect to t. Chain rules are also needed.

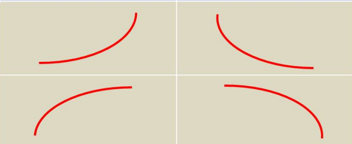
Properties of a Function

- Second derivative (L.21, Ch12.2):

$$d^2y/dx^2 = f''(x) = (f'(x))'$$

- What can derivatives tell (L.20-21, Ch12.1-2):

		Increasing	Decreasing
		f′(x) > 0	f′(x) < 0
Concave Upwards	f′′(x) >0		
Down-wards	f′′(x) <0		



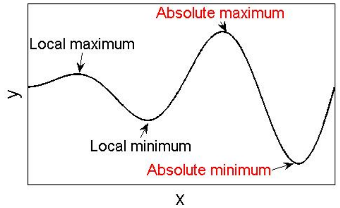
- Critical points at $x = c$ (L.20, Ch12.1): c is in the domain of f, $f'(c) = 0$ or not exist.
- Inflection points at $x = c$ (L.21, Ch12.2): c is in the domain of f,

$$f''(c) = 0 \text{ or does not exist.}$$

- Curve sketching: details in L.22 or Ch12.4.

Extrema and Optimization (Part 1)

Local and Absolute Extrema:



- Local Extrema:
  - only occurs at critical points (L.20, Ch12.1)
  - second derivative test (L.23, Ch12.5):

$$f'(c) = 0 \text{ and } \begin{cases} \text{(a) } f''(c) > 0 \Rightarrow \text{local minimum} \\ \text{(b) } f''(c) < 0 \Rightarrow \text{local maximum} \end{cases}$$

Extrema and Optimization (Part 2)

Local and Absolute Extrema:

- Absolute Extrema occur at critical points or end points(L.23, Ch12.5).
- One special case: the only critical point $\Rightarrow$ local $=$ absolute (L.23, Ch12.5).

Optimisation (details to be found in L.24 or Ch12.6):

1. Determine variables and the relationships among them
2. Mathematical modelling, the domain of definition for x may come from practice.
3. Find the absolute extrema
4. Interpretation.

Problems and Solutions

Example

$$f(x) = e^x(x^2 - 3).$$

(a) Find the derivative of $f(x)$ with respect to x

(b) Find the expression for the tangent line to $f(x)$ at $x = 0$

(c) Find the values of x, when the tangent lines are horizontal.

Problems and Solutions

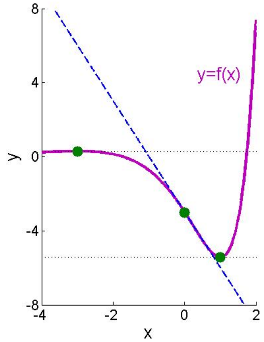
Solution

(a) By using the product rule

$$f'(x) = (e^x)'(x^2-3)+e^x(x^2-3)' = e^x(x^2+2x-3)$$

(b) The slope at $x = 0$ is $f'(0) = -3$, and it passes point $(0, f(0)) = (0, -3)$. So equation for the tangent line satisfies

$$\frac{y+3}{x-0} = -3 \Rightarrow 3x + y + 3 = 0.$$



(c) Since $e^x > 0$, $f'(x) = 0$ implies

$$x^2 + 2x - 3 = (x + 3)(x - 1) = 0.$$

Hence at $x = -3$ and $x = 1$, the tangent lines are horizontal.

Problems and Solutions

Example

Calculate the derivative of $f(x)$ with respect to x:

(a)

$$f(x) = \frac{x^2e^x}{\ln x}$$

(b)

$$f(x) = \frac{\sqrt{x}+5}{x^2}$$

(c)

$$f(x) = \sqrt{(2x-1)(x^2+1)}$$

(d)

$$f(x) = e^{(\ln x)^2}$$

Problems and Solutions

Solution - Part 1

(a)

$$f'(x) = \frac{(x^2e^x)'\ln x - (\ln x)'x^2e^x}{(\ln x)^2}.$$

From the product rule, we know $(x^2e^x)' = (x^2)'e^x + x^2(e^x)' = e^x(x^2 + 2x)$.

$$f'(x) = \frac{e^x(x^2+2x)\ln x - (1/x)x^2e^x}{(\ln x)^2} = \frac{e^x(x^2+2x)\ln x - xe^x}{(\ln x)^2}.$$

(b) Since

$$f(x) = x^{-3/2} + 5x^{-2},$$

we have

$$f'(x) = -\frac{3x^{-5/2}}{2} - 10x^{-3}.$$

Problems and Solutions

Solution - Part 2

(c) With the chain rule,

$$f'(x) = \frac{1}{2\sqrt{(2x-1)(x^2+1)}} \cdot ((2x-1)(x^2+1))'.$$

With the product rule,

$$((2x-1)(x^2+1))' = 2(x^2+1) + 2x(2x-1) = 6x^2 - 2x + 2.$$

Hence $f'(x) = \frac{3x^2-x+1}{\sqrt{(2x-1)(x^2+1)}}$.

(d) With the chain rule,

$$f'(x) = e^{(\ln x)^2}\cdot((\ln x)^2)' = e^{(\ln x)^2}\cdot 2\ln x\cdot(\ln x)' = \frac{2e^{(\ln x)^2}\ln x}{x}$$

Problems and Solutions

Example

Evaluate $\frac{dy}{dx}$ at $x = 0$ for

$$x \ln y = ye^x - 1 \qquad \text{(a)}.$$

Solution

$$(x \ln y)' = (ye^x - 1)'$$

$$\ln y + \frac{x}{y} \cdot \frac{dy}{dx} = e^x \cdot \frac{dy}{dx} + ye^x$$

Rearranging the above identity gives $\frac{dy}{dx} = \frac{\ln y - ye^x}{e^x - x/y}$.
At $x = 0$, from (a) we have $y = 1$. Thus

$$\left.\frac{dy}{dx}\right|_{(0,1)} = -1.$$

Problems and Solutions

Example

Peter is of height 1.75m, and he is walking away from a lamp post (street light) at a speed of 1m per second. He finds that his shadow is of the same length to his height when he is 3m away from the lamp post.

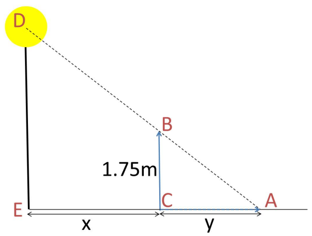
(a) What is the height of the lamp post?

(b) How fast is the top of the shadow moving?

Problems and Solutions

Solution

(a) The problem is set-up as shown in the right side. Then $x = 3$ and $y = 1.75$ (the same as Peter's height). Since Δ ABC is similar to Δ ADE,



$$\frac{y}{BC} = \frac{x+y}{DE} \Rightarrow DE = \frac{y+x}{y} \times BC = 4.75m$$

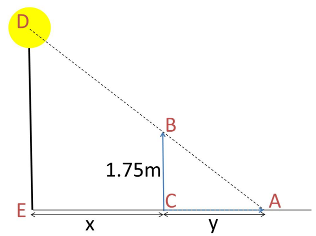
(b) we now know that $dx/dt = 1$. We can take the derivative on both sides of $y/1.75 = (x+y)/4.75$ with respect to t:

$$\frac{1}{1.75} \cdot \frac{dy}{dt} = \frac{1}{4.75} \cdot \left(\frac{dx}{dt} + \frac{dy}{dt}\right) \Rightarrow \frac{dy}{dt} = 0.583\text{m/s}.$$

Problems and Solutions

Solution

(b) we now know that $dx/dt = 1$. We can take the derivative on both sides of $y/1.75 = (x+y)/4.75$ with respect to t:



$$\frac{1}{1.75} \cdot \frac{dy}{dt} = \frac{1}{4.75} \cdot \left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dy}{dt} = 0.583\text{m/s}.$$

Since $s = x + y$ is the distance from the top of the shadow A to the point E. So the velocity of A moving is

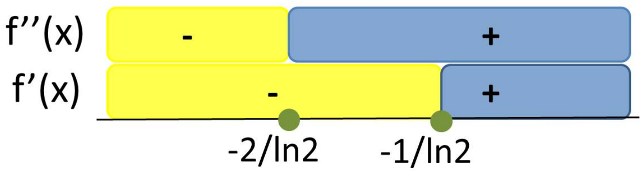
$$\frac{ds}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = 1.583\text{m/s}.$$

Example

Sketch $y = f(x) = x2^x$, where $x \in [-8, 1]$.

Solution - Part 1

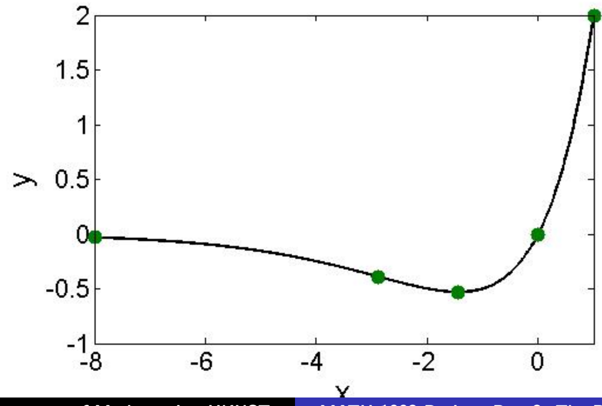
1. x lies between -8 and 1. At two boundaries, $f(-8) = -0.0313$ and $f(1) = 2$.
2. $f(x)$ is well-defined in $[-8, 1]$, no asymptotes.
3. $f'(x) = 2^x(1 + x\ln 2) \Rightarrow$ critical points: $x = -1/\ln 2$.
4. $f''(x) = 2^x((\ln 2)^2 x + 2\ln 2) \Rightarrow$ inflection points: $x = -2/\ln 2$.



Solution - Part 2

5. Evaluate $f(x)$ at all critical and inflection points:

x	-8	$-2/\ln 2$	$-1/\ln 2$	0	1
$f(x)$	-0.0313	-0.3905	-0.5307	0	2



Example

A 300-room hotel in Las Vegas is filled to capacity every night at $ 80 a room. For each $1 increase in rent, 3 fewer rooms are rented. If each rented room costs $10 to service per day, how much should the management charge for each room to maximise gross profit? What is the maximum gross profit?

Solution

Let x be number of room rented, then it is related to the price p by

$$3 \times (p - 80) = 300 - x \Rightarrow p = \frac{300 - x}{3} + 80.$$

Then the total profit = (price - service cost) $\times$ number, mathematically the problem becomes

$$\text{To maximise } F(x) = \left(\frac{300 - x}{3} + 80 - 10\right) x, \quad 0 \le x \le 300.$$

It is calculated that $F'(x) = 170 - \frac{2x}{3} \Rightarrow$ critical point: $x = 255$. It can be checked that $F''(x) < 0$, at $x = 255$ is the absolute maximum (the only local extremum). The price should be set to be $95 and the total profit is $ 21,675.00.