MATH 1003 Review: Part 2. Matrices

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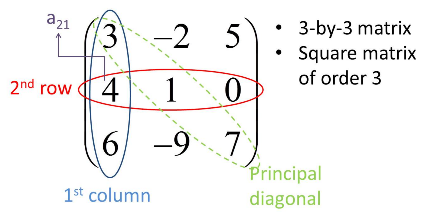
Matrices (Ch.4)

(i) System of linear equations in 2 variables (L.5, Ch4.1)

- Find solutions by graphing
- Supply and demand curve

(ii) Basic ideas about Matrices (L.6, Ch4.2)

- To know a matrix



- Row operation

$$R_i \leftrightarrow R_j, \quad kR_i \to R_i, \quad R_i + kR_j \to R_i$$

Matrices (Ch.4)

(iii) Gauss-Jordan Elimination (Method 1 to solve systems of linear eqns.)

- Find the corresponding augmented matrix (L.7, Ch4.3)
- Using row operation to get the reduced form (L.7, Ch4.3)
- Unique solution / no solution / infinite number of solutions (how many degrees of freedom) (L.7, Ch4.3)
- Application: variables and restrictions, positive numbers may be required for some real world problems (L.8, Ch4.3)

(iv) Matrix Operation (L.9, Ch4.4)

Operation		Input		Outcome	
Function	Expression	Input 1	Input 2	size	c_{ij} from
Add / Substraction	C = A ± B	$(a_{ij})_{m\times n}$	$(b_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	a_{ij}, b_{ij}
Scalar multiplication	C = kA	k	$(a_{ij})_{m\times n}$	$(c_{ij})_{m\times n}$	k, a_{ij}
Matrix Product	C = AB	$(a_{ij})_{m\times p}$	$(b_{ij})_{p\times n}$	$(c_{ij})_{m\times n}$	i-th row of A, j-th column of B

Matrices (Ch.4)

(v) Inverse Matrix (L.10, Ch4.5)

- Identity matrix I

$$IM = MI = M$$

- Given a square matrix M, its inverse matrix M^{-1} satisfies

$$MM^{-1} = M^{-1}M = I$$

- Find the inverse matrix M^{-1}:
  - ▷ Of order 2: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  - ▷ Of order 3: $(M|I) \xrightarrow{\text{Row operation}} (I|M^{-1})$
  - ▷ Not all matrices are invertible

(vi) Matrix Equation (L.11, Ch4.6)

- Solver to system of linear equations (method 2 to solve systems of linear eqns): if A is invertible,

$$AX = B \quad \Rightarrow \quad X = A^{-1}B.$$

(vii) Leontief input-output analysis (L.12, Ch.4.7)

- Technology matrix M for n-industry is constructed:

To produce		A	B	C	⋯
Need	A	m_{11}	m_{12}	m_{13}	⋯
	B	m_{21}	m_{22}	m_{23}	⋯
	C	m_{31}	m_{32}	m_{33}	⋯
	⋯	⋯	⋯	⋯	⋯

- The final demand matrix D and the output matrix X are $n \times 1$ matrices.
- The matrix equation for the model is

$$\underbrace{X}_{\text{output}} = \underbrace{M}_{\text{technology matrix}} X + \underbrace{D}_{\text{external demand}}.$$

- Therefore, $X = (I - M)^{-1}D$ if $I - M$ is invertible.

Example

Given

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -5 \\ -2 & -1 & 2 \end{pmatrix},$$

$C = AB$. Find the second column of matrix C.

Solution

A is of size 2×2 and B is of size 2×3, so C is of size 2×3. For its second column, we only need to calculate c_{12} and c_{22}.

- For c_{12}, the 1st row of A and the 2nd column of B are needed:

$$c_{12} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot 1 + 3 \cdot (-1) = -2.$$

- For c_{22}, the 2nd row of A and the 2nd column of B are needed:

$$c_{22} = \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \cdot 1 + 0 \cdot (-1) = 2.$$

The second column of C is $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

Example

Given a system of linear equations:

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 + 4x_3 = 5 \\ 2x_1 + hx_3 = k \end{cases}$$

(a) Write down the corresponding augmented matrix $(A|b)$ from $Ax = b$.

(b) For what value of h, the system can not have a unique solution (either no or infinite number). Is A invertible in this case?

(c) Then for what value of k, the system has infinite number of solution. Express the resulting solution.

Problems and Solutions

Solution

(a) The augmented matrix is $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 4 & 5 \\ 2 & 0 & h & k \end{array}\right]$

(b) To find the reduced form

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 4 & 5 \\ 2 & 0 & h & k \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & h-4 & k-2 \end{array}\right] \xrightarrow{R_2/2\to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & h-4 & k-2 \end{array}\right] \xrightarrow[-2R_2+R_3\to R_3]{R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & h-6 & k-6 \end{array}\right]$$

if $h = 6$, the last equation is $0 = k - 6$, where there is either infinite number of or no solutions. A is NOT invertible in this case.

Problems and Solutions

Solution

(c) To have infinite number of solution, $k = 6$, then the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

So the system of linear equations is transformed to

$$\begin{cases} x_1 + 3x_3 = 3 \\ x_2 + x_3 = 2 \end{cases}$$

If we set $x_3 = t$ as the free variable, the solution becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

Problems and Solutions

Example

Find the solution x_1, x_2 x_3 and x_4 for the following system:

$$\begin{cases} x_1 - x_3 + 2x_4 = 0 \\ 3x_1 - 4x_2 + 2x_3 = 13 \\ x_1 + x_2 - 3x_4 = -1 \\ 2x_1 + x_2 - x_3 - 5x_4 = -1 \end{cases}$$

Problems and Solutions

Solution - Part 1

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 3 & -4 & 2 & 0 & 13 \\ 1 & 1 & 0 & -3 & -1 \\ 2 & 1 & -1 & -5 & -1 \end{array}\right] \xrightarrow[\substack{-R_1+R_3\to R_3 \\ -2R_1+R_4\to R_4}]{-3R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -4 & 5 & -6 & 13 \\ 0 & 1 & 1 & -5 & -1 \\ 0 & 1 & 1 & -9 & -1 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -9 & -1 \\ 0 & 1 & 1 & -5 & -1 \\ 0 & -4 & 5 & -6 & 13 \end{array}\right] \xrightarrow[4R_2+R_4\to R_4]{-R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -9 & -1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 9 & -42 & 9 \end{array}\right]$$

Problems and Solutions

Solution - Part 2

$$\xrightarrow[\frac{R_4}{9}\to R_4]{\frac{R_3}{4}\to R_3}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -9 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{14}{3} & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -9 & -1 \\ 0 & 0 & 1 & -\frac{14}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{9R_4+R_2\to R_2 \\ -2R_4+R_1\to R_1}]{\frac{14}{3}R_4+R_3\to R_3}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Problems and Solutions

Solution - Part 2

$$\xrightarrow[R_3+R_1\to R_1]{-R_3+R_2\to R_2}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $x_1 = 1$, $x_2 = -2$, $x_3 = 1$, $x_4 = 0$.

Problems and Solutions

Example

Which of the following matrices is in reduced form?

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Answers may be found somewhere in the neighbouring slides

Problems and Solutions

Example

An economy is based on two sectors, energy (E) and water (W). To produced one dollar's worth of E requires 0.4 dollar's worth of E and 0.2 dollar's worth of W, and to produce one dollar's worth of W requires 0.1 dollaar's worth of E and 0.3 dollar's worth of W.

(a) Find the technology matrix M for the economy.

(b) Find the total output for each sector that is needed to satisfy a final demand of \$40 billion for energy and \$30 billion for water.

Answers to the question in the previous slide: B.

Solution

(a)

		To produce	
		E	W
Demand	E	0.4	0.1
	W	0.2	0.3

$$\Rightarrow M = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$$

(b) By setting x_1 - output for E and x_2 - output for W, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 40 \\ 30 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.1 \\ -0.2 & 0.7 \end{pmatrix}^{-1} \begin{pmatrix} 40 \\ 30 \end{pmatrix} = \begin{pmatrix} 77.5 \\ 65 \end{pmatrix}$$