Suggested Solutions.

HKUST

MATH1003 Calculus and Linear Algebra

Final Examination (Version A)

12th December 2012

12:30-15:30

Name:_

Student ID:_

Lecture Section:_

Directions:

- Do NOT open the exam until instructed to do so.
- Please turn off all phones and pagers, and remove headphones.
- Please write your name, ID number, and Lecture Section in the space provided above.
- On your MC answer sheet, please write your name and Lecture Section in the space provided. Write your ID number and fill the corresponding circles. On the blank space below "Date", write your exam paper version ("A" or "B").
- When instructed to open the exam, please check that you have 12 pages in addition to the cover page.
- Answer all questions. Show an appropriate amount of work for each problem. If you do not show enough work, you will get only partial credit.
- You may write on the backside of the pages, but if you use the backside, clearly indicate that you have done so.
- You may use an ordinary scientific calculator, but calculators with graphical, or symbolic calculation functions are NOT allowed.
- This is a closed book examination.
- **Cheating is a serious offense. Students caught cheating will receive a zero score, and will also be subjected to further penalties imposed by the University.**

Question No.	Points	Out of
Q. 1-20		40
Q. 21		10
Q. 22		18
Q. 23		10
Q. 24		10
Q. 25		12
Total Points		100

Part I: Multiple Choice Questions.
Each of the following MC questions is worth 2 points. No partial credit. Put your MC question answers in the MC answer sheet provided.

1. Kelly and Justin each win monetary prizes on a TV show. Kelly immediately deposits her money in an account paying 6.2% compounded annually, while Justin immediately deposits his in an account paying 5.1% compounded annually. After 7 years, they decide to get married, and discover that they have a total of $73955.41 combined in their two accounts. If the sum of their original prizes was $51,000, how much did Justin originally win, rounded to the nearest dollar?

   (a) $16,000 (b) $19,000 (c) $32,000 (d) $35,000 (e) None of the previous

   5

2. Suppose that you make an initial deposit of $22,000 and will make monthly withdrawal of $260, starting one month from now, followed by a final smaller payment (one month after the last regular payment). If the interest rat is 4.2% compounded monthly, how many payments will you receive, including the final smaller payment?

   (a) 101 (b) 100 (c) 99 (d) 98 (e) None of the previous

3. The following is the augmented matrix corresponding to a certain system of linear equations:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

   Which of the following statements must be true?

   (I) The augmented matrix is a reduced form. ✗
   (II) The corresponding system of linear equations has infinitely many solutions.
   (III) The corresponding system of linear equations has 2 free variables.

   (a) (I) only (b) (I) and (II) only (c) (II) only
   (d) (II) and (III) only (e) (I), (II) and (III)

4. Let A, B, C and X be 2×2 matrices such that $AX + BX = CX$, then which of the following statements must be true?

   (a) $A + B = C$ if $X \neq 0$.

   (b) $X = 0$ if $C = 0$.

   (c) $X = 0$ if $(A + B - C)^{-1}$ exists.

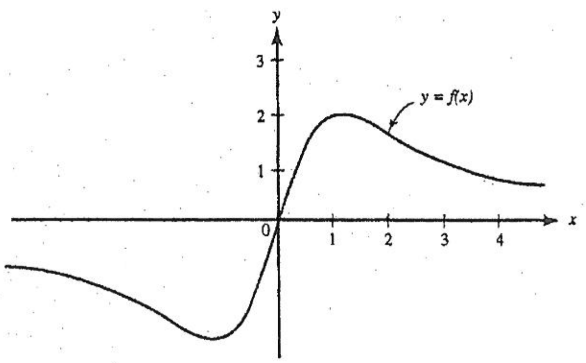
   (d) $A = C$ if $B = 0$.

   (e) None of the previous

5. If \$100 is invested at 6% compounded continuously, what amount will be in the account after 2 years, rounded to four decimal places?

   (a) \$112.0103 (b) \$112.9017 (c) \$102.7500 (d) \$112.7497 (e) None of the previous

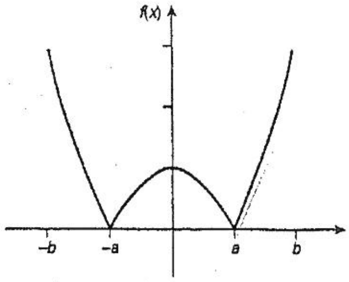
6. Which function could have the graph shown below?

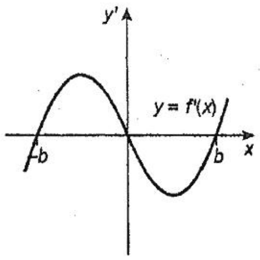


(a) $f(x) = \dfrac{x}{x^2+1}$ (b) $f(x) = \dfrac{2x}{x^2-1}$ (c) $f(x) = \dfrac{4x}{x+1}$

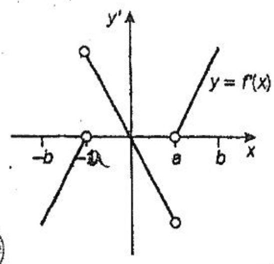
(d) $f(x) = \dfrac{4x}{x^2+1}$ (e) $f(x) = \dfrac{x^2+3}{x^2+1}$

7. The graph of $y = f(x)$ is shown below. Which of the following graphs could be the derivative?

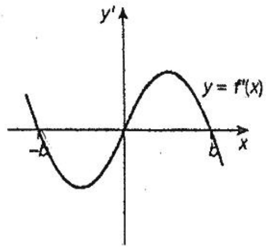




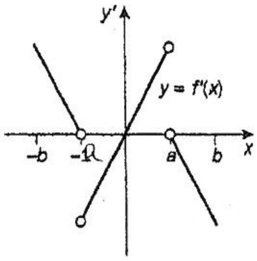
(a)



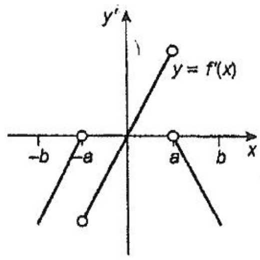
(b)



(c)



(d)



(e)

8. Suppose $ax^4 - by^2 = c^4$, where a, b and c are constants and $b \neq 0$. Find $\dfrac{dy}{dx}$

(a) $\dfrac{2ax^3 - 2c^3}{by}$ (b) $\sqrt{\dfrac{ax^4 - c^4}{b}}$ (c) $\dfrac{2ax^3}{by}$

(d) $\dfrac{2ax^3}{b}$ (e) None of the previous

9. If $1 + f(x) + x^2[f(x)]^3 = 0$ and $f(1) = 2$. Find $f'(1)$.

(a) $-\dfrac{16}{13}$ (b) $-\dfrac{1}{28}$ (c) $-\dfrac{1}{12}$ (d) $-\dfrac{1}{7}$ (e) None of the previous

10. If the sides of a square are growing at a constant rate of 5 cm/min, how fast is the area increasing when the area is 400 cm^2?

(a) 100 cm^2/min (b) 200 cm^2/min (c) 300 cm^2/min (d) 400 cm^2/min (e) 500 cm^2/min

11. A point is moving on the graph of $4x^2 + 9y^2e^y = 36$. When the point is at $(3, 0)$, its y-coordinate is decreasing by 2 units per second. How fast is its x-coordinate changing at that moment?

(a) Decreasing by 1 unit per second

(b) Increasing by 3 unit per second

(c) 0 unit per second

(d) Decreasing by 2 units per second

(e) None of the previous

12. Find $\lim_{x \to 3^+} \frac{x|3-x|}{x-3}$

(a) -3 (b) 3 (c) 0 (d) 1 (e) The limit does not exist

13. Given the function

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$$

Find $\lim_{x \to 2} f(x)$.

(a) 1 (b) 2 (c) 3 (d) 0 (e) The limit does not exist

14. Find the derivative of $f(x)$ at $x = 1$ where

$$f(x) = \frac{2x^5 - 4x^3 + 2x}{x^3}.$$

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 3 (e) None of the previous

15. Find $\frac{dy}{dt}$ at $t = 1$ where $y = (1 + e^t) \ln t$.

(a) $2e + 1$ (b) $e + 1$ (c) $2 + e^2$ (d) 0 (e) None of the previous

16. If $y = \ln(te^t)$, find $\frac{dy}{dt}$ at $t = 1$.

(a) $2e + 1$ (b) 2 (c) $1 + e$ (d) e^{-1} (e) None of the previous

17. If $f(x) = \sqrt[3]{x + 1} - e^{\frac{x}{3}}$, find $f''(0)$

(a) $\frac{1}{3}$ (b) $-\frac{2}{9}$ (c) $-\frac{1}{9}$ (d) 0 (e) $-\frac{1}{3}$

18. What is the slope of the tangent line to the curve $y = e^{x^2+1}$ at the point $(x, y) = (0, e)$?

(a) $3e$ (b) $2e$ (c) e (d) 0 (e) $-e$

19. Evaluate the following derivative

$$\frac{d}{du}\Big[\int [\, e^{u^2} + u^{-2}(1 + u)^3 \,]\, du \,\Big].$$

(a) $e^2 - 2u^{-3}$

(b) $e^{2u} - 6u^{-3}(1 + u)^2$

(c) $2ue^{u^2} - 6u^{-3}(1 + u)^2$

(d) $e^{u^2} + u^{-2}(1 + u)^3$

(e) None of the previous

20. Suppose that f and g are two continuous functions such that $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$, where $a < b$. Which of the following statements must be always true?

(a) $f(x) = g(x)$ for every x taken in interval $[a, b]$.

(b) $F(b) - F(a) = G(b) - G(a)$, where F is an antiderivative of f, G is an antiderivative of g.

(c) There must exist x in the interval $[a, b]$ such that $f(x) < g(x)$.

(d) There must exist x in the interval $[a, b]$ such that $f(x) > g(x)$.

(e) $\int_a^b |f(x)|\, dx = \int_a^b |g(x)|\, dx$

21

(a)

$$M = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.6 \end{bmatrix}$$

$$I-M = \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.4 \end{bmatrix}$$

$$(I-M)^{-1} = \frac{1}{0.6\times0.4-0.3\times0.5}\begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.6 \end{bmatrix}$$

$$= \frac{1}{0.09}\begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.6 \end{bmatrix} = \begin{bmatrix} \frac{40}{9} & \frac{30}{9} \\ \frac{50}{9} & \frac{60}{9} \end{bmatrix}$$

$$\begin{array}{r} 0.24 \\ \underline{0.15} \\ 0.09 \end{array}$$

$$\begin{bmatrix} \frac{40}{9} & \frac{30}{9} \\ \frac{50}{9} & \frac{60}{9} \end{bmatrix}\begin{bmatrix} 60 \\ 40 \end{bmatrix} = \begin{bmatrix} 400 \\ 600 \end{bmatrix}$$

$\therefore$ Total output of coal is 400, steel is 600.

(b). $60 \times 1.15 = 69, \quad 40 \times 1.3 = 52$

$$\begin{bmatrix} \frac{40}{9} & \frac{30}{9} \\ \frac{50}{9} & \frac{60}{9} \end{bmatrix}\begin{bmatrix} 69 \\ 52 \end{bmatrix} = \begin{bmatrix} 480 \\ 730 \end{bmatrix}$$

% increase in output of coal is $\frac{480-400}{400} = 20\%$

% increase on output of steel is $\frac{730-600}{600} = 21.67\%$

22. [*18 pts*] Given a function $f(x) = \dfrac{9(x^2-1)}{x^3}$, with first and second derivatives given as below:

$$f'(x) = \frac{9(3-x^2)}{x^4}, \qquad f''(x) = \frac{18(x^2-6)}{x^5}.$$

(a) Find the horizontal and vertical asymptotes of the graph of $y = f(x)$. [*2 pts*]

Vertical asymptotes at $x=0$.

Horizontal asymptotes at $y=0$

2

(b) Determine the interval of increase and interval of decrease of $f(x)$. [*3 pts*]

$f'(x) = 0$

$x = -\sqrt{3}, 0, \sqrt{3}$

$$\begin{array}{ccccccc} - & & + & & + & & - \\ & -\sqrt{3} & & 0 & & \sqrt{3} & \end{array}$$

Interval of increase: $(-\sqrt{3}, 0) \cup (0, \sqrt{3})$

Interval of decrease: $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

3

(c) Find all local maximum or local minimum of $f(x)$. Justify your answer. [*3 pts*]

Local maximum at $x = \sqrt{3}$ because $f''(\sqrt{3}) < 0$

Local minimum at $x = -\sqrt{3}$ because $f''(-\sqrt{3}) > 0$

3

(d) Find the concave upward interval and concave downward interval of the function. [3 pts]

$f''(x) = 0$

$x = -\sqrt{6}, 0, \sqrt{6}$

$-$, $+$, $-$, $+$

$-\sqrt{6}$ 0 $\sqrt{6}$

Concave up : $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$

Concave down : $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$

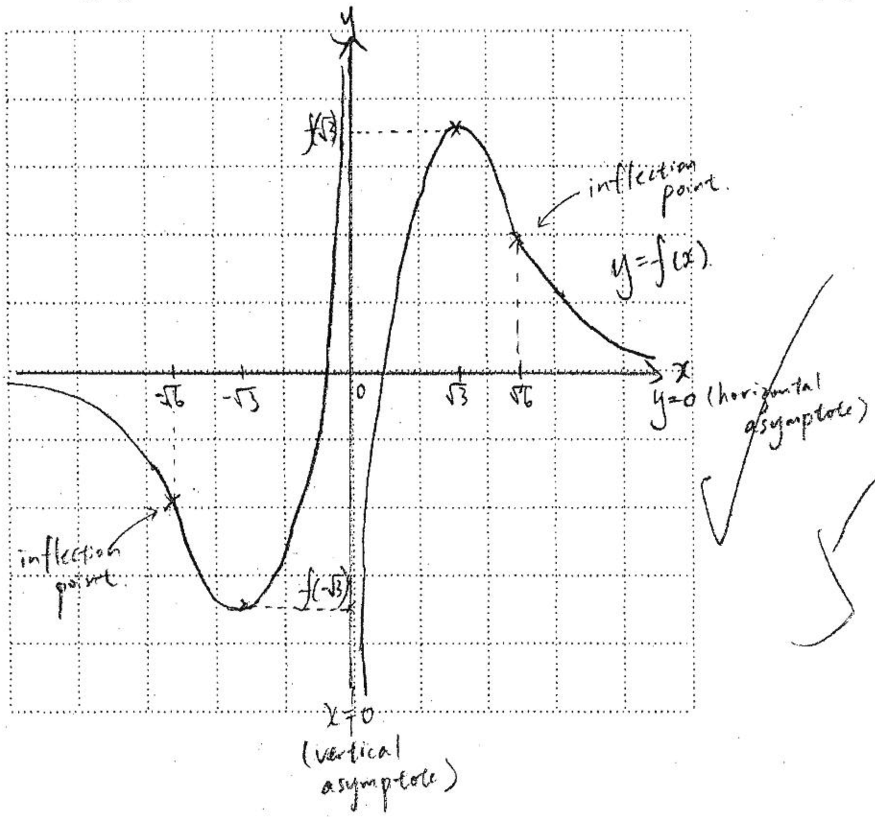
3

(d) Find all inflection point(s) of the function. [2 pts]

Inflection points : $-\sqrt{6}, \sqrt{6}$.

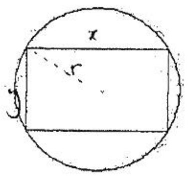
2

(e) Sketch the graph of the function in the given grid with appropriate scales on your axes, and include the asymptote of the graph found in part (a). Roughly indicate the positions of the inflection points on the graph. [5 pts]



5

23. [*10 pts*] The following figure is a rectangle inscribed in a circle of radius 4.



Area = xy

$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

(a) Write the total area of the rectangle as a function of one variable. [*4 pts*]

Let the width and the length of the rectangle be x and y respectively.

$$x^2+y^2=(2\cdot 4)^2.$$

$$y=\sqrt{64-x^2}$$

Area of the rectangle $= xy$

$$= x\sqrt{64-x^2}$$

(b) Find the dimensions of the rectangle with maximum area. Justify your answer. [*6 pts*]

Let the area of the rectangle be A.

$$A = x(64-x^2)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = -x^2(64-x^2)^{-\frac{1}{2}} + (64-x^2)^{\frac{1}{2}}$$

$$= (64-x^2)^{-\frac{1}{2}}(-x^2+64-x^2)$$

$$= (64-x^2)^{-\frac{1}{2}}(64-2x^2)$$

$$= 2(32-x^2)(64-x^2)^{-\frac{1}{2}}.$$

$$\frac{dA}{dx} = 0$$

$\therefore$ $x=-8$ (rej) or $x=-4\sqrt{2}$ (rej) or $x=4\sqrt{2}$ or $x=8$ (rej.) for $0<x<8$.

$$\underset{4\sqrt{2}.}{\overset{+ \quad\quad -}{\longrightarrow}}$$

$\therefore$ The area of the rectangle will be maximum at $x=4\sqrt{2}$.

When $x=4\sqrt{2}$, $y=\sqrt{64-(4\sqrt{2})^2}=4\sqrt{2}$.

$\therefore$ The dimension of the rectangle should be $(4\sqrt{2}\times 4\sqrt{2})$.

24. [*10 pts*] Evaluate the following indefinite integrals:

10

(a) $\int (e^x - \frac{2}{x} + 6x^5)\ dx$ [*2 pts*]

$\int (e^x - 2x^{-1} + 6x^5)\ dx$

$= e^x - 2\ln x + x^6 + C$

2

(b) $\int (2x^3 + x)(\sqrt{x^4 + x^2})\ dx$ (Hint: use substitution.) [*4 pts*]

$u = \sqrt{x^4 + x^2}$

$du = (x^4 + x^2)^{-\frac{1}{2}} (2x^3 + x)\ dx.$

$(2x^3 + x)\ dx = (x^4 + x^2)^{\frac{1}{2}}\ du.$

$\therefore \int (2x^3 + x)(x^4 + x^2)^{\frac{1}{2}}\ dx.$

$= \int u^2\ du$

$= \frac{1}{3} u^3$

$= \frac{1}{3}(x^4 + x^2)^{\frac{3}{2}} + C$

4

(c) $\int x\ln(2x)\ dx$ (Hint: use integration by parts formula $\int u\ dv = uv - \int v\ du$) [*4 pts*]

$u = \ln(2x)$ $du = \frac{1}{2x} \cdot 2\ dx$

$= \frac{1}{x}\ dx.$

$dv = x\ dx$ $v = \frac{1}{2}x^2$

$\int x \ln(2x)\ dx = \frac{1}{2}x^2 \ln(2x) - \int \frac{1}{2}x^2 \frac{1}{x}\ dx.$

$= \frac{1}{2}x^2 \ln(2x) - \frac{1}{2}\int x\ dx.$

$= \frac{1}{2}x^2 \ln(2x) - \frac{1}{4}x^2 + C$

4

25. (a) [*8 pts*] Let C_1 be the curve of equation: $y = f(x) = 5 - x^2$. Let C_2 be the curve of equation $y = g(x) = 2 - 2x$. It is known that C_1 and C_2 has two intersection points. Let R be the region, having finite area, bounded by the two curves C_1 and C_2.

(i) Find the x-coordinates and y-coordinates of the two intersection points. [*3 pts*]

$$5 - x^2 = 2 - 2x.$$

$$x = 3 \quad \text{or} \quad x = -1$$

$\therefore$ When $x = 3$, $y = -4$

When $x = -1$, $y = 4$

$\therefore$ The intersection points are $(3, -4)$ and $(-1, 4)$.

(ii) Find the area of the region R mentioned above. [*5 pts*]

$$\int_{-1}^{3} (5-x^2) - (2-2x) \, dx$$

$$= \int_{-1}^{3} -x^2 + 2x + 3 \, dx$$

$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3}$$

$$= \left(-\frac{1}{3}\cdot 27 + 9 + 3\cdot 3\right) - \left(-\frac{1}{3}(-1) + 1 + 3(-1)\right)$$

$$= 9 + \frac{5}{3}$$

$$= \frac{32}{3}$$

12

(b) [*4 pts*] Find the equation of the curve that passes through point $(1, 3)$ if the slope of the tangent line to the curve at various locations is given by

$$\frac{dy}{dx} = 12x^2 + 12x$$

$$y = \int 12x^2 + 12x \, dx$$

$$y = 4x^3 + 6x^2 + C$$

$\because$ The curve passes $(1, 3)$.

$$3 = 4 + 6 + C$$

$$C = -7$$

$\therefore$ The equation of the curve is

$$y = 4x^3 + 6x^2 - 7.$$

*** END OF PAPER ***