HKUST

MATH1003 Calculus and Linear Algebra

Final exam (Version A)

14th December 2016

12:30 - 14:30

S H Ho Sports Hall

Name:________________________

Student ID:________________________

Seat Number: ________________________

Lecture Section: ________________________

Directions:

- Do NOT open the exam until instructed to do so.
- Please turn off all phones and pagers, and remove headphones.
- Please write your name, student ID, Seat number and Lecture Section in the space provided above.
- When instructed to open the exam, please check that you have 10 pages in addition to the cover page.
- Answer all questions. Show an appropriate amount of work for each problem. If you do not show enough work, you will get only partial credit.
- Any forms of calculators are NOT allowed.
- This is a closed book examination.
- **Cheating is a serious offense. Students caught cheating will receive a zero score for the midterm exam, and will also be subjected to further penalties imposed by the University.**

Question No.	**Points**	**Out of**
Q. 1-9		45
Q. 9		20
Q. 10		20
Q. 11		20
Total Points		105

Part I: Answer the following multiple choice questions.

Put your MC question answers in CAPTICAL letters in the following boxes.

Question	1	2	3	4	5	Total
Answer	A	D	B	A	C	

Question	6	7	8	9	Total
Answer	C	A	C	D	

Each of the following MC questions is worth 5 points. No partial credit.

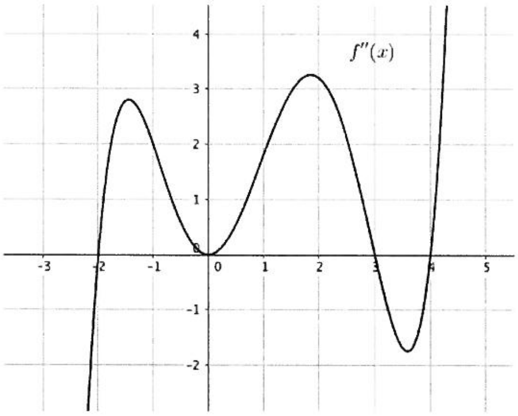
1. Air is pumped into a spherical balloon at the rate of 8 cubic centimeters per minute. What is the rate of change of the surface area per minute when the radius of the balloon is 2 centimeters? (The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$.)

   [✓ (a)] 8. (b) 8π. (c) 4π. (d) 2π. (e) 4.

$V = \frac{4}{3}\pi r^3$ $\quad \frac{dv}{dt} = \frac{4}{3}\pi 3 r^2 \frac{dr}{dt} = 8$ $\quad \frac{dr}{dt} = \frac{8}{4\pi r^2}$ $\quad \underline{r=2}$

$s = 4\pi r^2$ $\quad \frac{ds}{dt} = 4\pi 2 r \frac{dr}{dt}$ $\quad \frac{ds}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \frac{8}{4\pi r^2} = \frac{16}{r} = 8$

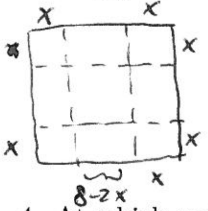
2. The following is a plot of $f''(x)$, the second derivative of a function $f(x)$. Find ALL the inflection points of $f(x)$.



(a) $x = 0, 3.5$. (b) $x = -1.5, 1.8$. (c) $x = -1.5, 0, 1.8, 3.5$. [✓ (d)] $x = -2, 3, 4$. (e) $x = -2, 0, 3, 4$.

3. A candy box is to be made out of a piece of cardboard that measures 8 by 8 inches. Squares of equal size will be cut out of each corner, and then the ends and sides will be folded up to form a rectangular box. What size square should be cut from each corner to obtain a maximum volume?

(a) 4. [✓ (b) $\frac{4}{3}$.] (c) $\frac{2}{3}$. (d) 2. (e) None of the above.



$V=(8-2x)^2 \cdot x$

$\frac{dv}{dx} = (8-2x)^2 + 2(8-2x)(-2)x$

$= (8-2x)(8-2x-4x) = (8-2x)(8-6x) = 0$

$8-6x=0$ $\boxed{x=\frac{4}{3}}$

4. At which point of x is the tangent line of the graph $y = e^{2x} - 2x + 1$ horizontal?

[✓ (a) $x = 0$.] (b) $x = \frac{\ln 2}{2}$ (c) $x = 1$ (d) $x = \frac{\ln 3}{2}$ (e) None of the above

$y' = e^{2x} \cdot 2 - 2 = 0$ $e^{2x} = 1 \Rightarrow 2x = 0$ $\underline{x = 0}$

5. What is $f''(0)$ for $f(x) = \ln(1 + e^x)$?

(a) 0. (b) $\frac{1}{2}$. [✓ (c) $\frac{1}{4}$.] (d) e. (e) None of the above

$f'(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$ $f''(x) = -\frac{0-(1+e^x)'}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$

$f''(0) = \frac{1}{(1+1)^2} = \frac{1}{4}$

6. Which of the following number is the slope of the tangent line to the curve given by

$$\ln(xy) = y^2 - 1$$

at the point $(x, y) = (1, 1)$?

(a) 0. (b) $\frac{1}{2}$. [✓ (c) 1.] (d) 2. (e) None of the above.

$\ln x + \ln y = y^2 - 1$ $x=1,\ y=1 \Rightarrow 1 + y' = 2y' \Rightarrow y' = 1$

$\frac{1}{x} + \frac{y'}{y} = 2y\,y'$

7. What value of A would make the function

$$f(x) = \begin{cases} Axe^{\frac{x}{2}} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a probability density function?

[✓ (a) $\frac{1}{4}$.] (b) 2. (c) $\frac{1}{2}$. (d) 4. (e) 1

$1 = \int_0^2 A\,x e^{\frac{x}{2}}\,dx = A\int_0^1 2y\,e^y\,2dy = 4A\int_0^1 y e^y dy = 4A\left(ye^y - \int e^y dy\right)$

$= 4A\left(ye^y - e^y\right)\Big|_0^1$

$\frac{x}{2} = y$ $x = 2y$

$dx = 2dy$

$\begin{cases} x=0 \rightsquigarrow y=0 \\ x=2 \rightsquigarrow y=1 \end{cases}$

$\begin{cases} u = y \\ dv = e^y dy \end{cases} \Rightarrow \begin{cases} du = dy \\ v = e^y \end{cases}$ $= 4A = 1$

$\Rightarrow \boxed{A = \frac{1}{4}}$

8. The shelf life (in years) of a laser pointer battery is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{2}{(x+2)^2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that a randomly selected laser pointer battery has a shelf life of from 1 to 4 years?

(a) $\frac{1}{4}$. (b) $\frac{1}{6}$. [✓ (c) $\frac{1}{3}$.] (d) $\frac{2}{5}$. (e) None of the above.

$$\int_1^4 \frac{2}{(x+2)^2}\,dx = 2\,\frac{(x+2)^{-2+1}}{-2+1}\Bigg|_1^4 = 2\left(-\frac{1}{(x+2)}\right)\Bigg|_1^4$$

$$= 2\left(\frac{1}{3}-\frac{1}{6}\right) = \frac{1}{3}$$

9. Which of the following is the value of the definite integral

$$\int_1^2 \ln\left(xe^{2x}\right)dx?$$

(a) $2\ln 2+1$. (b) $\ln 2+3$. (c) $\ln 2+1$. [✓ (d) $2\ln 2+2$.] (e) None of the above.

$$\int_1^2 \ln x + 2x\,dx = \int_1^2 \ln x\,dx + x^2\Big|_1^2 = \left(x\ln x - \int x\frac{1}{x}dx\right) + x^2\Big|_1^2$$

$$\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}dx \\ v = x \end{cases}$$

$$= (x\ln x - x + x^2)\Big|_1^2$$

$$= (2\ln 2 - 2 + 4) - (0 - 1 + 1)$$

$$= \boxed{2\ln 2 + 2}$$

Part II: Answer each of the following 3 long questions. Unless otherwise specified, numerical answers should be either exact or correct to 2 decimal places.

10. Consider the graph of the function $f(x) = \dfrac{x^2 + x + 2}{x - 1}$ (five sub-problems).

(1). What is the domain of $f(x)$? What are the vertical and horizontal asymptotes (if there are any)? What are the x- and y-intercepts (if there are any)?

domain: $x \neq 1$

$x = 0$: $f(0) = \frac{2}{-1} = -2$

$y = 0$: $x^2 + x + 2 = 0$. $\Delta = 1 - 4 \cdot 2 = -7 < 0$, no solution $\therefore$ $y \neq 0$

The vertical asymptotde is $x = 1$. No horizontal asymptote.

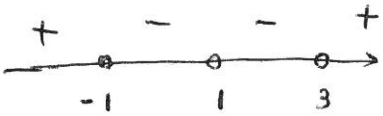
(2). List all critical numbers if there is any. Find the intervals on which $f(x)$ is increasing, and those on which $f(x)$ is decreasing.

$$f(x) = \frac{x^2 - x + 2x - 2 + 4}{x - 1} = x + 2 + \frac{4}{(x-1)}$$

$$f'(x) = 1 - \frac{4}{(x-1)^2} = \frac{(x-1)^2 - 4}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2} = 0$$

$x = 3$, $x = -1$ are critical numbers ($x = 1$ is not as it is not in the domain)

sign chart of $f'(x)$



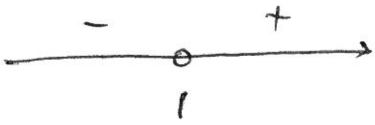
increasing: $(-\infty, -1) \cup (3, \infty)$

decreasing: $(-1, 1) \cup (1, 3)$

(3). List all inflection points if there is any. Find the intervals on which $f(x)$ is concave upward, and those on which $f(x)$ is concave downward.

$$f''(x) = \left(1 - 4(x-1)^{-2}\right)' = (-4)(-2)(x-1)^{-3} = \frac{8}{(x-1)^3}$$

$f''(x)$ sign chart :



concave down : $(-\infty, 1)$

concave up : $(1, \infty)$

no inflection points.

(4) Find the local maximum and local minimum of $y = f(x)$. Are they absolute maximum and absolute minimum of $y = f(x)$? Why?

By the 1st or 2nd derivative test,

$x = 3$ is a local min

$x = -1$ is a local max. They are not absolute max/min.

(5) Use the above information to sketch the graph $y = f(x)$.

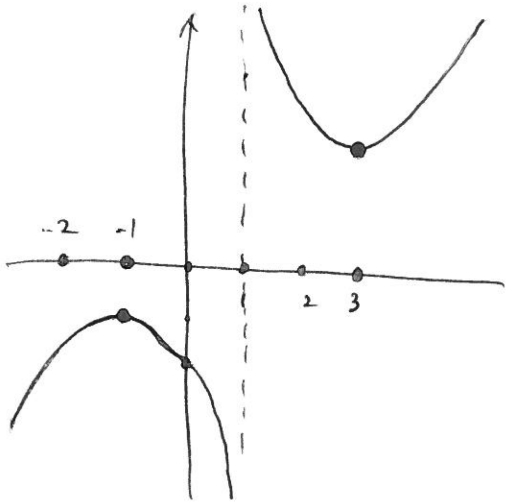
$y = f(x) = \dfrac{x^2 + x + 2}{x - 1}$

$x = 2$, $y = \dfrac{4+2+2}{1} = 8$

$x = 3$: $y = \dfrac{9+3+2}{2} = 7$

$x = -1$: $y = \dfrac{1-1+2}{-2} = -1$

$x = 0$; $y = -2$



11. Calculate the indicated integrations (four sub-problems)

(1).

$$\int \left(x^3 + \frac{1}{x} + e^x\right) dx.$$

$$\underline{\frac{x^4}{4} + \ln|x| + e^x + C \, .}$$

(2).

$$\int x \left(e^x + e^{x^2}\right) dx.$$

$$\int x e^x dx + \int x e^{x^2} dx = A + B$$

$$A : \begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases} \quad A = x e^x - \int e^x dx = x e^x - e^x + C$$

$$B : \begin{cases} u = x^2 \\ du = 2x dx \end{cases} \Rightarrow B = \int \frac{1}{2} e^u du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c$$

$$x dx = \frac{1}{2} du$$

$$\therefore A + B = \underline{x e^x - e^x + \frac{1}{2} e^{x^2} + c}$$

(3).

$$\int \left(\ln x + \frac{1}{x} \right) dx.$$

$$\underline{I} = \int \ln x \, dx + \ln|x| + C \quad - \quad A = \int \ln x \, dx$$

$$\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x} dx \\ v = x \end{cases} \qquad A = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

$$\therefore \underline{I} = x \ln x - x + \ln|x| + C = \underline{x \ln x - x + \ln x + C}$$

(4).

$$\tilde{A} = \int \left(\ln x + \frac{1}{x} \right)^2 dx.$$

$$\tilde{A} = \int (\ln x)^2 dx + 2 \int \frac{\ln x}{x} dx + \int \frac{1}{x^2} dx = A_1 + A_2 + A_3$$

$$A_1 : \begin{cases} u = (\ln x)^2 \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = 2 (\ln x) \frac{1}{x} dx \\ v = x \end{cases}$$

$$A_1 = x (\ln x)^2 - \int x \, 2 (\ln x) \frac{1}{x} dx = x (\ln x)^2 - 2 \int \ln x \, dx$$

$$= x (\ln x)^2 - 2A \overset{\text{from above}}{=} x (\ln x)^2 - 2 (x \ln x - x) + C \,.$$

$$A_2 : \begin{cases} u = \ln x \\ du = \frac{1}{x} dx \end{cases} \qquad A_2 = 2 \int u \, du = u^2 + C = (\ln x)^2 + C$$

$$A_3 = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} + C = - \frac{1}{x} + C$$

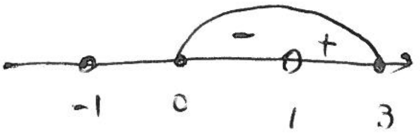
$$\text{so } \tilde{A} = A_1 + A_2 + A_3 = \underline{x (\ln x)^2 - 2 (x \ln x - x) + (\ln x)^2 - \frac{1}{x} + C}$$

12. Set-up the integral for computation

Instruction: just set-up the integral without explicitly computing it. For example, the area bounded by $y = x$ and the x axis over the interval $[1, 2]$ is given by $\int_1^2 x dx$. No need to compute it.

(1). Find the area between the graph of $f(x) = x^2 - 1$ and the x axis over the interval $[0, 3]$.

$$x^2-1=(x-1)(x+1)$$



$$\text{Area} = \int_0^1 -(x^2-1)\,dx$$

$$+ \int_1^3 x^2-1\,dx .$$

(2). Find the area bounded by the graphs of $f(x) = x^2 - 1, g(x) = -x - 3, x = -1$ and $x = 2$.

$x^2-1-(-x-3) = x^2+x+2 > 0$ for all x as $\Delta = 1-4\times 2 = -7 <$

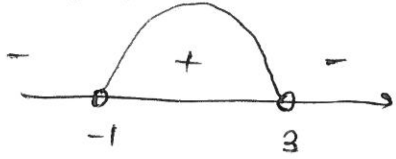
$$\text{so Area} = \int_{-1}^2 x^2+x+2\,dx .$$

(3). Find the area of the finite region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.

$f(x) \cdot g(x) = 5 - x^2 - (2-2x) = -x^2 + 2x + 3$

$= -(x+1)(x-3) = 0$

$x = -1$, $x = 3$. sign chart of $f(x) - g(x)$



so Area $= \int_{-1}^{3} 5 - x^2 - (2-2x)\,dx$

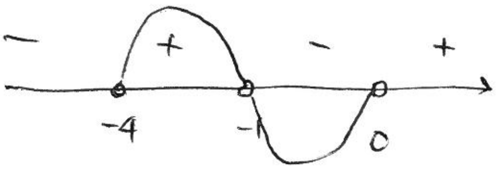
$= \int_{-1}^{3} -x^2 + 2x + 3\,dx$

(4). Find the area of the finite region bounded by the graphs of $f(x) = x^3 + 5x^2 + 5x$ and $g(x) = x$.

$h(x) = f(x) - g(x) = x(x^2 + 5x + 4) = x(x+1)(x+4) = 0$

$x = 0, -1, -4$

sign chart: of $h(x)$



The area is $\int_{-4}^{-1} x(x+1)(x+4)\,dx + \int_{-1}^{0} -x(x+1)(x+4)\,dx$.