PAIR CORRELATION OF SUMS OF RATIONALS WITH BOUNDED HEIGHT

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ABSTRACT. For each positive integer Q, let $\mathscr{F}_Q$ denote the Farey sequence of order Q. We prove the existence of the pair correlation measure associated to the sum $\mathscr{F}_Q + \mathscr{F}_Q$ modulo 1, as Q tends to infinity, and compute the corresponding limiting pair correlation function.

1. Introduction

The study of local spacings, which measure the distribution of a sequence in a more subtle way than the classical Weyl uniform distribution ([46]), was initiated by physicists (see Wigner [47] and Dyson [11], [12], [13]), in order to understand the spectra of high energies. These notions have received a great deal of attention in many areas of mathematical physics, analysis, probability theory and number theory. In most cases of interest in number theory it is very challenging to prove the existence of the limiting spacing measures. Many such sequences are predicted to have a Poisson distribution, and some important results of this type are due to Hooley [19], [20] on residue classes relatively prime with a large modulus q, Gallagher [17] on gaps between primes, Sarnak [39] on values at integers of binary quadratic forms, and Rudnick and Sarnak on pair correlation of fractional parts of polynomials [34]. Further results have been obtained by a number of authors. Primitive roots modulo p were studied in [9], and the distribution of visible points from the origin in dilations of a region Ω was established in [4]. The spacing distribution of fractional parts of lacunary sequences has been obtained by Rudnick and one of the authors in [36] and [38] (see also [7]), and the distribution of small powers of a primitive root was studied in [37]. Boca and one of the authors [6] investigated the pair correlation of values of rational functions modulo p. Kurlberg and Rudnick [27] (see also [26]) established the distribution of squares modulo highly composite integers. The spacings between the energy levels of a two-dimensional harmonic oscillator (see Pandey, Bohigas and Giannoni [32] and Bleher [2],[3]) are essentially those between the numbers $\alpha n \pmod 1$, where the gaps take at most three values (see Sós [40] and Swierczkowski [41]). The distribution of energy levels of a boxed oscillator reduces to that of $\alpha n^2 \pmod 1$, which is conjectured to be Poissonian (see Berry and Tabor [1]). Rudnick, Sarnak and one of the authors [35] (see also [50]) proved that this conjecture holds true for a large class of numbers α satisfying certain Diophantine conditions. Eigenvalues on

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multidimensional flat tori, and values at integers of homogeneous polynomials, were studied by Vanderkam [42], [43], [44]. Correlation densities of inhomogeneous quadratic forms were investigated by Marklof [28], [29]. The distribution of fractional part of $\sqrt{n}$ was established by Elkies and McMullen [14]. The distribution of imaginary parts of zeros of primitive L-functions is believed to be the same as the GUE distribution studied by Random Matrix Theory. Important work in this area was done by Montgomery [30], Rudnick and Sarnak [33], and Katz and Sarnak [24], see also [25]. One striking difference between the GUE model and the Poissonian model is that the density function vanishes at the origin in the GUE case but not in the Poissonian case. For this reason, it is said that in the Poissonian case one has "level clustering" while in the GUE case one has "level repulsion". Here the word "level", coined by physicists, refers to the possibly infinitely many stages of a process. One has an even stronger repulsion in the context described below.

Here we investigate a new type of question, which concerns two different notions: the pair correlation of the given sequence and the natural additive structure of the ambient space. More specifically, for each positive integer Q let $\mathscr{F}_Q$ denote the Farey sequence of order Q (for basic properties of the Farey sequence see [18]), as the Qth level of our process, that is, the set of all rationals in $[0,1]$ of height bounded by Q (the height of a rational number, in irreducible form, is defined to be the maximum of the absolute values of its numerator and denominator). The pair correlation measure associated to $\mathscr{F}_Q$ was proved to converge, as $Q \to \infty$, by Boca and one of the authors [8]. They showed that the limiting measure is absolutely continuous with respect to the Lebesgue measure, and provided an explicit formula for the corresponding limiting pair correlation function $g(\lambda)$,

$$g(\lambda) = \frac{6}{\pi^2\lambda^2} \sum_{1 \le k \le \frac{\pi^2\lambda}{3}} \varphi(k) \log \frac{\pi^2\lambda}{3k}\,,$$

for any $\lambda > 0$, where φ is Euler's totient function.

Let $\mathscr{F}_Q = \{\frac{a}{b} \ : \ 1 \le a \le b \le Q,\ (a,b) = 1\}$ be the set of Farey fractions of order Q and also let $\mathscr{F}_Q + \mathscr{F}_Q \subset [0,1)$ denote the set of all sums of pairs of fractions in $\mathscr{F}_Q$ written modulo 1. Our goal is to understand whether addition of Farey fractions influences the pair correlation measure. For this purpose, we compare the pair correlation of $\mathscr{F}_Q + \mathscr{F}_Q \pmod 1$ with that of $\mathscr{F}_Q$, as $Q \to \infty$. From a technical point of view the pair correlation measure of the sum $\mathscr{F}_Q + \mathscr{F}_Q$ is more difficult to handle than that of $\mathscr{F}_Q$. The Weil bounds [45], [15] for Kloosterman sums, which played a decisive role in [8], fail to solve the problem. A natural strategy would be to employ Deligne bounds [10] for two dimensional hyper-Kloosterman sums, but the range of the sums turns out to be too short for this method to succeed either. Karatsuba [21], [22], [23] devised a method for bounding certain exponential sums, and Friedlander and Iwaniec applied it successfully in [16], but our short ranges are outside the scope of this method either. As pointed out in [48], [49], one sometimes obtains more cancellation by averaging the pair correlations themselves rather than by averaging their expressions in terms of exponential sums. Inspired by this idea, we adjust our use of exponential sums, and barely obtain enough cancellation to complete the proof. In order to state our main result, we introduce a multiplicative arithmetic function ψ, which plays a similar role for $\mathscr{F}_Q + \mathscr{F}_Q$ to the one played by Euler's function for $\mathscr{F}_Q$. We define ψ in terms of its associated Dirichlet series,

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta^4(s)} \prod_{p \text{ prime}} H_p(s), \tag{1}$$

where $\zeta(s)$ is the Riemann Zeta function, and $H_p(s)$ is given by

$$H_p(s) = 1 + \frac{(p-1)(p+4)}{p(p+3)} \left\{ -\frac{4}{p-1} + \sum_{k=1}^{\infty} \frac{\left(1-p^{(k+1)(1-s)}\right)^4 - \left(1-p^{k(1-s)}\right)^4}{(1-p^{1-s})p^{k(2-s)}} \right\}.$$

Theorem 1. *The limiting pair correlation function of $\mathscr{F}_Q + \mathscr{F}_Q$ modulo 1 exists, as $Q \to \infty$, on any subinterval $\mathbf{I} \subset [0,1]$, and is given by*

$$g_2(\lambda) = \frac{c}{\pi^2 \lambda^2} \sum_{1 \le k \le \frac{\pi^4 \lambda}{9}} \psi(k) \log^3 \frac{\pi^4 \lambda}{9k},$$

for any $\lambda > 0$, where

$$c = \prod_{p \text{ prime}} \left(1 - \frac{2}{p(p+1)}\right)\left(1 - \frac{3}{p(p+2)}\right).$$

The above functions $g(\lambda)$ and $g_2(\lambda)$ being distinct, we see that addition of Farey fractions does influence, in this sense, the pair correlation. Their graphs are shown in Figure 1, together with $g_{GUE}(\lambda) = 1 - \left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2$ and $g_{\text{Poisson}} =$ constant equal to 1.

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2. A Uniform distribution result

Let $\mathscr{F}_Q = \{\gamma_1, \ldots, \gamma_{N(Q)}\}$ denote the Farey sequence of order Q with $1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1$ and $\mathscr{F} = (\mathscr{F}_Q)_Q$. Let $x_{ij} \equiv \gamma_i + \gamma_j \pmod 1$ and denote by the set $\mathcal{G}_Q := \{x_{ij} : 1 \le i, j \le N(Q)\} = \mathscr{F}_Q + \mathscr{F}_Q \pmod 1$ counting multiplicities. The sequence of sequences $\mathcal{G} = (\mathcal{G}_Q)_{Q \in \mathbb{N}}$ is uniformly distributed along the unit interval. More precisely,

Lemma 1. *For any subinterval $\mathbf{I} \subset [0,1]$, denote $\mathcal{G}_{\mathbf{I}}(Q) := \mathcal{G}_Q \bigcap \mathbf{I}$. Then*

$$\#\mathcal{G}_{\mathbf{I}}(Q) = \frac{9|\mathbf{I}|}{\pi^4} Q^4 + O\left(Q^3 (\log Q)^{3/2}\right).$$

Our method actually gives a more general counting result. For any continuously differentiable function $f : R^k \to R$ with compact support, we define

$$Df = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots., \frac{\partial f}{\partial x_k} \right\rangle$$

and

$$||Df||_\infty = \sum_{j=1}^{k} \left|\left| \frac{\partial f}{\partial x_j} \right|\right|_\infty.$$

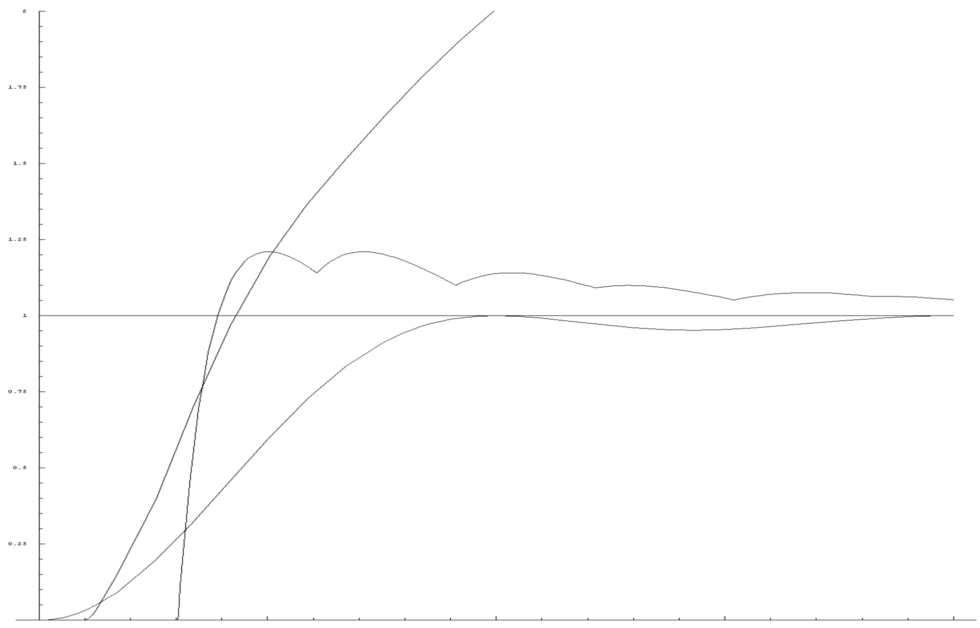

FIGURE 1. Graphs of $g(\lambda)$, $g_2(\lambda)$, $g_{GUE}(\lambda)$ and g_{Poisson}

Lemma 2. *Suppose* $\mathbf{I}$ *is a finite interval,* $G \in C^1(\mathbb{R})$ *with Supp* $(G) \subset \mathbf{I}$. *Define*

$$g(y) = \sum_{n\in\mathbb{Z}} G(y+n), \quad S_{Q,G} = \sum_{\gamma,\gamma' \in \mathscr{F}_Q} g(\gamma + \gamma').$$

Then

$$S_{Q,G} = \left(\int_{\mathbf{I}} G(x)\mathrm{d}x \right) \frac{9Q^4}{\pi^4} + E_{Q,G},$$

where

$$E_{Q,G} \ll Q^2(\log Q)^3 ||DG||_\infty |\mathbf{I}| + Q^3 \log Q \left| \int_{\mathbf{I}} G(x)\mathrm{d}x \right|.$$

Proof of Lemma 2: For $y \in \mathbb{R}$, let

$$g(y) = \sum_{n\in\mathbb{Z}} a_n e(ny)$$

be the Fourier series expansion of g. Then

$$\begin{aligned} S_{Q,G} &= \sum_{\gamma,\gamma'\in\mathscr{F}_Q}\sum_{n\in\mathbb{Z}} a_n e\left(n(\gamma+\gamma')\right) = \sum_{n\in\mathbb{Z}} a_n \sum_{\gamma\in\mathscr{F}_Q} e(n\gamma) \sum_{\gamma'\in\mathscr{F}_Q} e(n\gamma') \\ &= \sum_{n\in\mathbb{Z}} a_n \left(\sum_{\substack{1\le d\le Q\\ d|n}} dM\left(\frac{Q}{d}\right)\right)^2 \\ &= \sum_{1\le d_1,d_2\le Q} d_1 d_2 M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right) \sum_{l\in\mathbb{Z}} a_{[d_1,d_2]l}. \end{aligned}$$

Consider for each $y>0$ the function

$$G_y(x) = \frac{1}{y} G\left(\frac{x}{y}\right), \quad x\in\mathbb{R}.$$

By properties of the Fourier transform,

$$\widehat{G}_y(x) = \widehat{G}(yx),$$

and using Poisson summation formula,

$$\sum_{l\in\mathbb{Z}} a_{[d_1,d_2]l} = \sum_{l\in\mathbb{Z}} \widehat{G}([d_1,d_2]l) = \sum_{l\in\mathbb{Z}} \widehat{G}_{[d_1,d_2]}(l) = \sum_{l\in\mathbb{Z}} G_{[d_1,d_2]}(l).$$

Thus

$$S_{Q,G} = \sum_{1\le d_1,d_2\le Q} d_1 d_2 M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right) \sum_{l\in\mathbb{Z}} \frac{1}{[d_1,d_2]} G\left(\frac{l}{[d_1,d_2]}\right).$$

Applying Lemma 8 of [5], we obtain

$$\sum_{l\in\mathbb{Z}} \frac{1}{[d_1,d_2]} G\left(\frac{l}{[d_1,d_2]}\right) = \int_{\mathbf{I}} G(x)\mathrm{d}x + O\left(||DG||_\infty\left(\frac{|\mathbf{I}|}{[d_1,d_2]} + \frac{2}{[d_1,d_2]^2}\right)\right).$$

Therefore,

$$\begin{aligned} S_{Q,G} &= \left(\int_{\mathbf{I}} G(x)\mathrm{d}x\right)\left(\sum_{1\le d_1,d_2\le Q} d_1 d_2 M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right)\right) + E_{G,1} \\ &= \left(\int_{\mathbf{I}} G(x)\mathrm{d}x\right)\left(\sum_{1\le d\le Q} dM\left(\frac{Q}{d}\right)\right)^2 + E_{G,1}, \end{aligned}$$

where

$$E_{G,1} \ll \sum_{1\le d_1,d_2\le Q} Q^2 \left(||DG||_\infty\left(\frac{|\mathbf{I}|}{[d_1,d_2]} + \frac{2}{[d_1,d_2]^2}\right)\right).$$

Since

$$\begin{aligned}\sum_{1\le d_1,d_2\le Q}\frac{1}{[d_1,d_2]} &= \sum_{1\le\delta\le Q}\sum_{\substack{1\le d_1,d_2\le Q\\(d_1,d_2)=\delta}}\frac{1}{[d_1,d_2]}=\sum_{1\le\delta\le Q}\sum_{\substack{1\le q_1,q_2\le Q\\(q_1,q_2)=1}}\frac{1}{\delta q_1q_2}\\ &\ll (\log Q)^3,\end{aligned}$$

and

$$\sum_{1\le d_1,d_2\le Q}\frac{1}{[d_1,d_2]^2}\le\sum_{1\le\delta\le Q}\sum_{\substack{1\le q_1,q_2\le Q\\(q_1,q_2)=1}}\frac{1}{\delta^2q_1^2q_2^2}\le\left(\sum_{1\le\delta\le Q}\frac{1}{\delta^2}\right)^3=O(1),$$

it follows that,

$$E_{G,1}\ll Q^2(\log Q)^3\,||DG||_\infty|\mathbf{I}|.$$

Moreover we observe that

$$\begin{aligned}\sum_{d\le Q}dM\left(\frac{Q}{d}\right) &= \sum_{d\le Q}d\sum_{r\le Q/d}\mu(r)=\sum_{r\le Q}\mu(r)\sum_{d\le Q/r}d\\ &= \sum_{r\le Q}\mu(r)\left(\frac{Q}{r}+O(1)\right)^2\\ &= \frac{Q^2}{2}\sum_{r\le Q}\frac{\mu(r)}{r^2}+O(Q\log Q),\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{d\le Q}dM\left(\frac{Q}{d}\right) &= \frac{Q^2}{2}\left(\frac{6}{\pi^2}+O\left(\frac{1}{Q}\right)\right)+O(Q\log Q)\\ &= \frac{3Q^2}{\pi^2}+O(Q\log Q),\end{aligned}$$

which finally gives

$$\left(\sum_{1\le d\le Q}dM\left(\frac{Q}{d}\right)\right)^2=\frac{9Q^4}{\pi^4}+O(Q^3\log Q).$$

Combining all these estimates completes the proof of Lemma 2. ∎

Proof of Lemma 1. We will approximate the characteristic function $\chi_{\mathbf{I}}$ of $\mathbf{I}$ by a C^1 function. To this end, consider the function $f(x)=3x^2-2x^3$ for $x\in[0,1]$. First note the following properties:

- $f'(x)=6x(1-x)\ge 0$ and $|f'(x)|\le 3/2$ for $x\in[0,1]$;
- $f'(0)=f'(1)=0, f(0)=0, f(1)=1$;
- $\int_0^1 f(x)\mathrm{d}x=1/2$.

For real numbers $a < b < c < d$, we define the function $g_{a,b,c,d} : \mathbb{R} \longrightarrow [0,1]$ by

$$g_{a,b,c,d}(t) = \begin{cases} 0 & : \quad t \le a; \\ f\left(\frac{t-a}{b-a}\right) & : \quad a < t \le b; \\ 1 & : \quad b < t \le c; \\ f\left(1 - \frac{t-c}{d-c}\right) & : \quad c < t \le d; \\ 0 & : \quad d < t. \end{cases}$$

It is easy to see that $g_{a,b,c,d} \in C^1(\mathbb{R})$ with Supp $(g_{a,b,c,d}) \subset [a,d]$, and

$$||Dg_{a,b,c,d}||_\infty \le \frac{3}{2} \max\left(\frac{1}{b-a}, \frac{1}{d-c}\right),$$

$$\int_{\mathbb{R}} g_{a,b,c,d}(x)\mathrm{d}x = c - b + \frac{b-a}{2} + \frac{d-c}{2}.$$

Now let $G = \chi_{[a,b]}$, the characteristic function of interval $\mathbf{I} = [a,b] \subset [0,1]$. Putting $a_1 = a - \epsilon, a_2 = a + \epsilon, b_1 = b + \epsilon, b_2 = b - \epsilon$ and $G_1 = g_{a_1,a,b,b_1}, G_2 = g_{a,a_2,b_2,b}$, we may denote by

$$f(y) = \sum_{n\in\mathbb{Z}} G(y+n), \;\; f_1(y) = \sum_{n\in\mathbb{Z}} G_1(y+n)$$

and $f_2(y) = \sum_{n\in\mathbb{Z}} G_2(y+n)$, to obtain that

$$S_{Q,G} = \sum_{\gamma,\gamma'\in\mathscr{F}_Q} f(\gamma+\gamma'), S_{Q,G_1} = \sum_{\gamma,\gamma'\in\mathscr{F}_Q} f_1(\gamma+\gamma'), S_{Q,G_2} = \sum_{\gamma,\gamma'\in\mathscr{F}_Q} f_2(\gamma+\gamma').$$

Since $G_2 \le G = \chi_{[a,b]} \le G_1$, we have

$$S_{Q,G_2} \le S_{Q,G} \le S_{Q,G_1}.$$

Noticing that

$$\int_{\mathbb{R}} G_1(x)\mathrm{d}x = b - a + \epsilon = |\mathbf{I}| + \epsilon, \quad \int_{\mathbb{R}} G_2(x)\mathrm{d}x = b - a - \epsilon = |\mathbf{I}| - \epsilon,$$

and

$$||DG_1||_\infty \le \frac{3}{\epsilon}, \quad ||DG_2||_\infty \le \frac{3}{\epsilon},$$

we may use lemma 2 to obtain

$$\begin{aligned} S_{Q,G_1} &= (|\mathbf{I}| + \epsilon)\frac{9}{\pi^4}Q^4 + E_{Q,G_1}, \\ S_{Q,G_2} &= (|\mathbf{I}| - \epsilon)\frac{9}{\pi^4}Q^4 + E_{Q,G_2}, \end{aligned}$$

where

$$E_{Q,G_1} \ll Q^2 (\log Q)^3 \frac{3}{\epsilon}(|\mathbf{I}| + 2\epsilon) + Q^3 \log Q(|\mathbf{I}| + \epsilon),$$

and

$$E_{Q,G_2} \ll Q^2 (\log Q)^3 \frac{3}{\epsilon}|\mathbf{I}| + Q^3 \log Q(|\mathbf{I}| - \epsilon).$$

Choosing

$$\epsilon = \frac{(\log Q)^{3/2}}{Q},$$

we have

$$S_{Q,G_1} = S_{Q,G_2} = \frac{9|\mathbf{I}|}{\pi^4}Q^4 + E,$$

where

$$E \ll Q^3(\log Q)^{3/2},$$

Therefore

$$\#\mathcal{G}_{\mathbf{I}}(Q) = S_{Q,G} = \frac{9|\mathbf{I}|}{\pi^4}Q^4 + O\left(Q^3(\log Q)^{3/2}\right),$$

which completes the proof of Lemma 1. ∎

3. Pair correlation of sums of Farey fractions

For each positive integer Q, let $\mathscr{F}_Q = \{\gamma_1, \ldots, \gamma_{N(Q)}\}$ denote the Farey sequence of order Q with $1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1$. Let $\mathbf{I}$ be a subinterval of $[0,1]$. Denote by $x_{ij} \equiv \gamma_i + \gamma_j \pmod 1$ and $\mathcal{G}_Q := \{x_{ij} : 1 \le i, j \le N(Q)\} = \mathscr{F}_Q + \mathscr{F}_Q \pmod 1$, the set of sum of Farey sequences of order Q counted with multiplicity, $\mathcal{G} = (\mathcal{G}_Q)_Q$ and $\mathcal{G}_{\mathbf{I}}(Q) := \mathcal{G}_Q \bigcap \mathbf{I}$. Let $\#\mathcal{G}_{\mathbf{I}}(Q)$ be the cardinality of $\mathcal{G}_{\mathbf{I}}(Q)$. It is known from Lemma 1 that

$$N = \frac{\#\mathcal{G}_{\mathbf{I}}(Q)}{|\mathbf{I}|} = \frac{9Q^4}{\pi^4} + O(Q^3(\log Q)^{3/2}). \tag{2}$$

Our goal is to estimate the quantity

$$S_{Q,\mathbf{I}}(\wedge) := \#\left\{(x,y) \in \mathcal{G}_{\mathbf{I}}(Q) \times \mathcal{G}_{\mathbf{I}}(Q) : \begin{array}{c} x \neq y, \\ x - y \in \frac{(0,\wedge)}{N} + \mathbb{Z} \end{array}\right\}$$

for any positive real number $\wedge$ as $Q \to \infty$. In fact we prove a more general result.

Lemma 3. *Given the functions $G, H \in C^1(\mathbb{R})$ with Supp $(G) \subset (0,1)$ and Supp $(H) \subset (0,\wedge)$ for some $\wedge > 0$, define*

$$h(y) = \sum_{n \in \mathbb{Z}} H(N(y+n)), \quad g(y) = \sum_{n\in\mathbb{Z}} G(y+n),$$

and let

$$S_{Q,\mathbf{I},H,G} = \sum_{x,y \in \mathcal{G}_Q} h(x-y)g(x)g(y).$$

Then we have

$$S_{Q,\mathbf{I},H,G} = \frac{9Q^4}{\pi^4}\left(\int_0^1 G(z)^2\,\mathrm{d}z\right)\int_0^{\wedge} H(x)g_2(x)\,\mathrm{d}x + E_{Q,\mathbf{I},H,G},$$

where for any $x > 0$,

$$g_2(x) = \frac{c}{\pi^2 x^2}\sum_{1 \le k \le \frac{\pi^4 x}{9}} \psi(k)\log^3\frac{\pi^4 x}{9k}. \tag{3}$$

Here

$$c := \prod_{p \ prime} \left(1 - \frac{2}{p(p+1)}\right)\left(1 - \frac{3}{p(p+2)}\right),$$

ψ is the multiplicative function defined in (1) *and for any $\eta > 0$,*

$$E_{Q,\mathbf{I},H,G} \ll_{\mathbf{I},H,G,\eta} Q^{4-\frac{1}{18}+\eta}.$$

Note that assuming Lemma 3 with the error term $Q^{4-\frac{1}{18}+\eta}$, for $0 < \eta < \frac{1}{18}$, we may obtain

$$\begin{aligned} \lim_{Q\to\infty} \frac{S_{Q,\mathbf{I},H,G}}{\#\mathcal{G}_{\mathbf{I}}(Q)} &= \lim_{Q\to\infty} \frac{S_{Q,\mathbf{I},H,G}}{\frac{9|\mathbf{I}|}{\pi^4}Q^4} \\ &= \frac{\int_0^1 G(z)^2 \mathrm{d}z}{|\mathbf{I}|} \int_0^{\wedge} H(x) g_2(x)\, \mathrm{d}x. \end{aligned}$$

Let the smooth function G approach $\chi_{\mathbf{I}}$, the characteristic function of the interval $\mathbf{I}$, so that

$$\frac{\int_0^1 G(z)^2 \mathrm{d}z}{|\mathbf{I}|} \to 1.$$

Also let the smooth function H approach $\chi_{(0,\wedge)}$, the characteristic function of the interval $(0, \wedge)$. By a standard approximation argument, we see that the pair correlation function of $\mathcal{G}$ along the subinterval $\mathbf{I}$ of $[0, 1]$ exists and is independent of the location and length of the subinterval. This completes the proof of Theorem 1.

Proof of Lemma 3. The proof of lemma 3 will require several steps. Throughout the proof, all constants implied by the big "O" or "$\ll$" notation may depend on the functions H and G.

3.1. **Fourier series expansion and Poisson summation formula.** If the Fourier series expansion of the functions h and g are given by

$$h(y) = \sum_{n\in\mathbb{Z}} c_n e(ny)$$

and

$$g(y) = \sum_{n\in\mathbb{Z}} a_n e(ny)$$

for $y \in \mathbb{R}$, then it follows that

$$\begin{aligned} S_{Q,\mathbf{I},H,G} &= \sum_{\gamma_1,\gamma_2,\gamma_1',\gamma_2' \in \mathscr{F}_Q} \sum_m c_m e(m(\gamma_1+\gamma_2) - m(\gamma_1'+\gamma_2')) \times \\ &\quad \sum_n a_n e(n(\gamma_1+\gamma_2)) \sum_r a_r e(r(\gamma_1'+\gamma_2')) \\ &= \sum_{m,n,r} c_m a_n a_r \sum_{\gamma_1 \in \mathscr{F}_Q} e((m+n)\gamma_1) \sum_{\gamma_2 \in \mathscr{F}_Q} e((m+n)\gamma_2) \times \\ &\quad \sum_{\gamma_1' \in \mathscr{F}_Q} e((r-m)\gamma_1') \sum_{\gamma_2' \in \mathscr{F}_Q} e((r-m)\gamma_2') \\ &= \sum_{m,n,r} c_m a_n a_r \left(\sum_{\gamma \in \mathscr{F}_Q} e((m+n)\gamma) \right)^2 \left(\sum_{\gamma \in \mathscr{F}_Q} e((r-m)\gamma) \right)^2 . \end{aligned}$$

Therefore

$$S_{Q,\mathbf{I},H,G} = \sum_{m,n,r} c_m a_n a_r \left(\sum_{\substack{1 \le d \le Q, \\ d \mid m+n}} dM\left(\frac{Q}{d}\right) \right)^2 \left(\sum_{\substack{1 \le d \le Q, \\ d \mid r-m}} dM\left(\frac{Q}{d}\right) \right)^2 .$$

Changing the summation indices using $m+n=m'$, $r-m=n'$, $m=r'$, we have $m=r'$, $n=m'-r'$, $r=n'+r'$. Consequently in terms of m', n', r', we have

$$\begin{aligned} S_{Q,\mathbf{I},H,G} &= \sum_{m',n',r'} c_{r'} a_{m'-r'} a_{n'+r'} \left(\sum_{\substack{1 \le d \le Q, \\ d \mid m'}} dM\left(\frac{Q}{d}\right) \right)^2 \left(\sum_{\substack{1 \le d \le Q, \\ d \mid n'}} dM\left(\frac{Q}{d}\right) \right)^2 \\ &= \sum_{1 \le d_1,d_2,d_3,d_4 \le Q} d_1 \cdots d_4 M\left(\frac{Q}{d_1}\right) \cdots M\left(\frac{Q}{d_4}\right) \sum_{\substack{r,m,n \in \mathbb{Z}, \\ d_1 \mid m, d_2 \mid m, \\ d_3 \mid n, d_4 \mid n,}} c_r a_{m-r} a_{n+r} . \end{aligned}$$

Using an argument similar to that from [8] with Poisson summation formula, the inner sum on r, m, n is given as

$$\begin{aligned}
&\sum_{r\in\mathbb{Z}} c_r \sum_{m\in\mathbb{Z}} a_{[d_1,d_2]m-r} \sum_{n\in\mathbb{Z}} a_{[d_3,d_4]n+r} \\
= &\sum_{m,n} \frac{G\left(\frac{m}{[d_1,d_2]}\right) G\left(\frac{n}{[d_3,d_4]}\right)}{[d_1,d_2][d_3,d_4]} \sum_r c_r e\left(\left(\frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right) r\right) \\
= &\sum_{m,n} \frac{G\left(\frac{m}{[d_1,d_2]}\right) G\left(\frac{n}{[d_3,d_4]}\right)}{[d_1,d_2][d_3,d_4]} \sum_r H\left(N\left(r + \frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right).
\end{aligned}$$

3.2. **Further Reductions.** We need several reductions to convert the expression of $S_{Q,\mathbf{I},H,G}$ to a manageable form.

3.2.1. **First Reduction.** First of all, note that since Supp $(G) \subset (0,1)$, Supp $(H) \subset (0,\wedge)$, $[d_1,d_2], [d_3,d_4] \le Q^2$ and $N \sim \frac{9}{\pi^4}Q^4$, if $r \ne 0$, then as Q is sufficiently large, we have $H\left(N\left(r + \frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right) = 0$. Hence we may assume that $r = 0$.

3.2.2. **Second Reduction.** For positive integers d_1, d_2, d_3, d_4, let $\widetilde{u} = ([d_1,d_2],[d_3,d_4])$ and

$$e_1 = \frac{[d_1,d_2]}{\widetilde{u}}, \quad e_2 = \frac{[d_3,d_4]}{\widetilde{u}}.$$

Since $(e_1, e_2) = 1$, there is a unique integer $\widetilde{a}_2$ such that $0 < \widetilde{a}_2 < e_1, \widetilde{a}_2 e_2 \equiv 1 \pmod{e_1}$. Choose $\widetilde{a}_1 = (1 - \widetilde{a}_2 e_2)/e_1$, so that $\widetilde{a}_1 e_1 + \widetilde{a}_2 e_2 = 1$. Changing the summation indices with $m' = e_2 m - e_1 n,\ n' = \widetilde{a}_1 m + \widetilde{a}_2 n$, we have $m = \widetilde{a}_2 m' + e_1 n',\ n = -\widetilde{a}_1 m' + e_2 n'$, and hence

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} G\left(\frac{m}{[d_1,d_2]}\right) G\left(\frac{n}{[d_3,d_4]}\right) H\left(N\left(\frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right) = \\
&\sum_{m',n'\in\mathbb{Z}} G\left(\frac{\widetilde{a}_2 m'}{[d_1,d_2]} + \frac{n'}{\widetilde{u}}\right) G\left(\frac{-\widetilde{a}_1 m'}{[d_3,d_4]} + \frac{n'}{\widetilde{u}}\right) H\left(\frac{N m'}{[d_1,d_2,d_3,d_4]}\right).
\end{aligned}$$

Using

$$M\left(\frac{Q}{d}\right) = \sum_{r\le Q/d} \mu(r),$$

and changing the order of summation we rewrite $S_{Q,\mathbf{I},H,G}$ as

$$\begin{aligned}
S_{Q,\mathbf{I},H,G} = &\sum_{1\le r_1,r_2,r_3,r_4\le Q} \mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4) \sum_{\substack{d_1\le Q/r_1,\\ \vdots \\ d_4\le Q/r_4,}} \frac{d_1d_2d_3d_4}{[d_1,d_2][d_3,d_4]} \times \\
&\sum_{m,n\in\mathbb{Z}} G\left(\frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right) G\left(\frac{-\widetilde{a}_1 m}{[d_3,d_4]} + \frac{n}{\widetilde{u}}\right) H\left(\frac{Nm}{[d_1,d_2,d_3,d_4]}\right).
\end{aligned}$$

3.2.3. **Third Reduction.** For positive integers d_1, d_2, d_3, d_4, denote

$$\delta = \frac{d_1 d_2 d_3 d_4}{[d_1, d_2, d_3, d_4]}.$$

If Q is sufficiently large, then

$$\frac{9Q^4}{\pi^4}(1-\epsilon) < N < \frac{9Q^4}{\pi^4}(1+\epsilon),$$

for any $0 < \epsilon < 1$. Since Supp $(H) \subset (0, \wedge)$, to have a non-zero contribution from H, we need

$$\begin{aligned} 0 < mr_1r_2r_3r_4\frac{9}{\pi^4}(1-\epsilon)\cdot\delta &< mr_1r_2r_3r_4\frac{N}{Q^4}\cdot\delta \\ &\leq mr_1r_2r_3r_4\frac{N}{r_1r_2r_3r_4d_1d_2d_3d_4}\cdot\delta \\ &= \frac{Nm}{[d_1, d_2, d_3, d_4]} < \wedge, \end{aligned}$$

which reduces to the condition

$$mr_1r_2r_3r_4\delta < \frac{\pi^4\wedge}{9(1-\epsilon)}.$$

Denoting

$$C_\wedge = \frac{\pi^4\wedge}{9}, \tag{4}$$

and choosing ϵ sufficiently small, we have

$$1 \leq mr_1r_2r_3r_4\delta \leq C_\wedge.$$

3.2.4. **Fourth Reduction.** Fix m, r_1, r_2, r_3, r_4 and $\delta = \frac{d_1d_2d_3d_4}{[d_1,d_2,d_3,d_4]}$ bounded by $C_\wedge$. Since $\widetilde{u}|\delta$, and Supp $(G) \subset [0,1]$, to have a non-zero contribution from G, we need

$$0 < \frac{\widetilde{a}_2 m}{[d_1, d_2]} + \frac{n}{\widetilde{u}} < 1.$$

There are only finitely many integers n satisfying this inequality. Denote by $\mathcal{A}$ the finite set consisting of all possible values of such n. Changing the order of summation we obtain that

$$\begin{aligned} S_{Q,\mathbf{I},H,G} = & \sum_{\substack{mr_1r_2r_3r_4\delta\leq C_\wedge,\\ n\in\mathcal{A},}} \mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)\delta \sum_{\substack{d_i\leq Q/r_i, 1\leq i\leq 4,\\ \frac{d_1d_2d_3d_4}{[d_1,d_2,d_3,d_4]}=\delta,\\ ([d_1,d_2],[d_3,d_4])=\widetilde{u},}} \frac{1}{\widetilde{u}} \times \\ & G\left(\frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right) G\left(\frac{-\widetilde{a}_1 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right) H\left(\frac{Nm\delta}{d_1d_2d_3d_4}\right). \end{aligned}$$

3.2.5. **Fifth Reduction.** Since $\widetilde{a}_1 e_1 + \widetilde{a}_2 e_2 = 1$, we have

$$\frac{\widetilde{a}_1}{[d_3,d_4]} + \frac{\widetilde{a}_2}{[d_1,d_2]} = \frac{\widetilde{u}}{[d_1,d_2]\cdot[d_3,d_4]}\ ,$$

and it follows that

$$\left|\frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}} - \left(\frac{-\widetilde{a}_1 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right)\right| = \left|\frac{\widetilde{a}_2 m}{[d_1,d_2]} - \frac{-\widetilde{a}_1 m}{[d_1,d_2]}\right|$$
$$= \frac{mu}{[d_1,d_2]\cdot[d_3,d_4]} = \frac{m}{[d_1,d_2,d_3,d_4]} = \frac{m\delta}{d_1d_2d_3d_4} \le \frac{C_\wedge}{d_1d_2d_3d_4}\ ,$$

and

$$G\left(\frac{-\widetilde{a}_1 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right) = G\left(\frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right) + O\left(\frac{1}{d_1d_2d_3d_4}\right).$$

As a result of this reduction, we get

$$S_{Q,\mathbf{I},H,G} = \sum_{\substack{mr_1r_2r_3r_4\delta\le C_\wedge,\\ n\in\mathcal{A},}} \mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)\delta\cdot\sum + E_0,$$

where the inner sum is given by

$$\sum = \sum_{\substack{d_i\le Q/r_i, 1\le i\le 4,\\ \frac{d_1d_2d_3d_4}{[d_1,d_2,d_3,d_4]}=\delta,\\ ([d_1,d_2],[d_3,d_4])=\widetilde{u},}} \frac{1}{\widetilde{u}} G\left(\frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}}\right)^2 H\left(\frac{Nm\delta}{d_1d_2d_3d_4}\right). \tag{5}$$

Moreover the error term E_0 can be estimated as

$$E_0 \ll \sum_{1\le d_1,d_2,d_3,d_4\le Q,} \frac{1}{\widetilde{u}\cdot d_1d_2d_3d_4} \ll (\log Q)^4 \ll_\eta Q^\eta\,, \tag{6}$$

for any $\eta > 0$.

3.2.6. **Sixth Reduction.** Fix integers $m, r_1, r_2, r_3, r_4, \delta, n$ and u. Define

$$P_\delta = \{a\in\mathbb{N}:\ \text{for any prime } p, p|a \implies p|\delta\}. \tag{7}$$

For positive integers d_1, d_2, d_3, d_4, $d_i \le Q/r_i$, $1\le i\le 4$ with $\frac{d_1d_2d_3d_4}{[d_1,d_2,d_3,d_4]}$ and $([d_1,d_2],[d_3,d_4]) = \widetilde{u}$, factoring d_1,d_2,d_3,d_4 as $d_1 = a_1q_1, d_2 = a_2q_2, d_3 = a_3q_3, d_4 = a_4q_4$ with $a_i \in P_\delta$ and $(q_i,\delta) = 1$ for $1\le i\le 4$ together with $[d_1,d_2] = [a_1,a_2]q_1q_2$, $\widetilde{u} = ([d_1,d_2],[d_3,d_4]) = ([a_1,a_2],[a_3,a_4])$, and $\frac{d_1d_2d_3d_4}{[d_1,d_2,d_3,d_4]} = \delta$ implies that

$$\frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]} = \delta,\quad \frac{q_1q_2q_3q_4}{[q_1,q_2,q_3,q_4]} = 1.$$

Here $(q_i, q_j) = 1$ for $i \neq j$. Using these observations we can rewrite $\sum$ in (5) as

$$\sum = \sum_{\substack{a_i \le Q/r_i, 1\le i\le 4,\\ a_1,a_2,a_3,a_4 \in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,\\ ([a_1,a_2],[a_3,a_4])=\widetilde{u},}} \frac{1}{\widetilde{u}} \cdot \sum\nolimits_1, \tag{8}$$

where the inner sum in (8) is given by

$$\sum\nolimits_1 = \sum_{\substack{q_i \le Q/a_ir_i, 1\le i\le 4,\\ (q_i,q_j)=1, i\neq j,\\ (q_i,\delta)=1,}} G\left(\frac{1}{\widetilde{u}}\left(\frac{\widetilde{a}_2 m}{\frac{[a_1,a_2]q_1q_2}{\widetilde{u}}} + n\right)\right)^2 H\left(\frac{Nm\delta}{a_1a_2a_3a_4q_1q_2q_3q_4}\right). \tag{9}$$

3.2.7. **Seventh Reduction.** Next fix positive integers $a_1, a_2, a_3, a_4 \in P_\delta$. Let

$$a = \frac{[a_1,a_2]}{\widetilde{u}}, \quad b = \frac{[a_3,a_4]}{\widetilde{u}}, \text{ so that } (a,b) = 1 \text{ and } a|\delta, b|\delta, \widetilde{u}|\delta. \tag{10}$$

Define the functions

$$f(x) = G\left(\frac{1}{\widetilde{u}} \cdot (mx+n)\right)^2, \ h(x,y,z,w) = H\left(\frac{N\lambda}{xyzw}\right), \tag{11}$$

where

$$\lambda = \frac{m\delta}{a_1a_2a_3a_4} \neq 0. \tag{12}$$

We have

$$\begin{aligned} e_1 &= \frac{[d_1,d_2]}{\widetilde{u}} = \frac{[a_1,a_2]q_1q_2}{\widetilde{u}} = aq_1q_2 \le Q^2, \\ e_2 &= \frac{[d_3,d_4]}{\widetilde{u}} = \frac{[a_3,a_4]q_3q_4}{\widetilde{u}} = bq_3q_4 \le Q^2, \end{aligned}$$

and

$$0 < \widetilde{a}_2 < aq_1q_2, \quad \widetilde{a}_2(bq_3q_4) \equiv 1 \pmod{aq_1q_2}. \tag{13}$$

Denoting

$$\delta_i = a_ir_i \ge 1, \quad 1 \le i \le 4, \tag{14}$$

we can rewrite (9) in the form

$$\sum\nolimits_1 = \sum_{\substack{q_i \le Q/\delta_i, 1\le i\le 4,\\ (q_i,q_j)=1, i\neq j,\\ (q_i,\delta)=1,}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) h(q_1,q_2,q_3,q_4). \tag{15}$$

3.3. **Further Estimations.** We will need some further estimations in several stages.

3.3.1. **First Step.** We know that

$$||f||_\infty = O(1), \quad ||Df||_\infty = O(1). \tag{16}$$

Choosing $0 < \epsilon < 1/2$, one has

$$\frac{9Q^4}{2\pi^4} < \frac{9Q^4}{\pi^4}(1-\epsilon) < N < \frac{9Q^4}{\pi^4}(1+\epsilon) < \frac{27Q^4}{2\pi^4},$$

for Q sufficiently large. Since Supp $(H) \subset (0, \wedge)$, and $h(x,y,z,w) \neq 0$ for $0 < x \leq Q/\delta_1, 0 < y \leq Q/\delta_2, 0 < z \leq Q/\delta_3$ and $0 < w \leq Q/\delta_4$, we must have that $0 < \frac{N\lambda}{xyzw} < \wedge$. This implies that

$$\frac{Q}{\delta_1} \geq x > \frac{N\lambda}{\wedge yzw} \geq \frac{\frac{9Q^4 m\delta}{2\pi^4 a_1a_2a_3a_4}}{\frac{Q^3}{a_2r_2a_3r_3a_4r_4}\wedge} = \frac{9m\delta r_1r_2r_3r_4}{2\pi^4\wedge}\frac{Q}{\delta_1}.$$

Similar lower bounds can be obtained for y, z, w too. Denoting

$$c_\wedge = \frac{9m\delta r_1r_2r_3r_4}{2\pi^4\wedge}, \tag{17}$$

we have

$$h(x,y,z,w) \neq 0 \quad \Longrightarrow \quad \begin{array}{c} c_\wedge \cdot Q/\delta_1 \leq x \leq Q/\delta_1, \\ c_\wedge \cdot Q/\delta_2 \leq y \leq Q/\delta_2, \\ c_\wedge \cdot Q/\delta_3 \leq z \leq Q/\delta_3, \\ c_\wedge \cdot Q/\delta_4 \leq w \leq Q/\delta_4. \end{array} \tag{18}$$

Next for the function h, using

$$||h||_\infty = O(1), \tag{19}$$

and

$$\left|\frac{\partial h}{\partial x}(x,y,z,w)\right| = \left|H'\left(\frac{N\lambda}{xyzw}\right)\right| \cdot \frac{N\lambda}{xyzw} \cdot \frac{1}{x},$$

from (18), we obtain that

$$\left|\frac{\partial h}{\partial x}(x,y,z,w)\right| \leq ||DH||_\infty \cdot \wedge \cdot \frac{\delta_1}{c_\wedge Q} \ll \frac{\delta_1}{Q}. \tag{20}$$

Similarly,

$$\left|\frac{\partial h}{\partial y}(x,y,z,w)\right| \ll \frac{\delta_2}{Q} \tag{21}$$

$$\left|\frac{\partial h}{\partial z}(x,y,z,w)\right| \ll \frac{\delta_3}{Q} \tag{22}$$

$$\left|\frac{\partial h}{\partial w}(x,y,z,w)\right| \ll \frac{\delta_4}{Q}. \tag{23}$$

3.3.2. **Second Step.** We rewrite (15) in the form

$$\sum\nolimits_1 = \sum_{\substack{q_1\le Q/\delta_1,\\ q_2\le Q/\delta_2,\\ (q_1,q_2\delta)=1,\\ (q_2,\delta)=1,}} \sum_{\substack{q_3\le Q/\delta_3,\\ q_4\le Q/\delta_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) h(q_1,q_2,q_3,q_4). \tag{24}$$

Fixing q_1, q_2, we may denote the inner sum in (24) as

$$\sum\nolimits_2 = \sum_{\substack{q_3\le Q/\delta_3,\\ q_4\le Q/\delta_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) h(q_1,q_2,q_3,q_4). \tag{25}$$

Let K_3, K_4 be large positive integers to be chosen later and T_3, T_4 be real numbers such that

$$T_3K_3 = \frac{Q}{\delta_3}, \quad T_4K_4 = \frac{Q}{\delta_4}.$$

Therefore (25) becomes

$$\sum\nolimits_2 = \sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} \sum_{\substack{(k_3-1)T_3<q_3\le k_3T_3,\\ (k_4-1)T_4<q_4\le k_4T_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) h(q_1,q_2,q_3,q_4). \tag{26}$$

Since $(k_3-1)T_3 < q_3 \le k_3T_3 = \frac{Q}{\delta_3}\frac{k_3}{K_3}$, $(k_4-1)T_4 < q_4 \le k_4T_4 = \frac{Q}{\delta_4}\frac{k_4}{K_4}$, using (22) and (23) we have

$$\left|h(q_1,q_2,q_3,q_4) - h\left(q_1,q_2,\frac{Q}{\delta_3}\frac{k_3}{K_3},\frac{Q}{\delta_4}\frac{k_4}{K_4}\right)\right| \ll \frac{(\delta_3+\delta_4)(T_3+T_4)}{Q}.$$

Inserting this into (26), we deduce that

$$\sum\nolimits_2 = \sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} h\left(q_1,q_2,\frac{Q}{\delta_3}\frac{k_3}{K_3},\frac{Q}{\delta_4}\frac{k_4}{K_4}\right) \sum_{\substack{(k_3-1)T_3<q_3\le k_3T_3,\\ (k_4-1)T_4<q_4\le k_4T_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) + E_2', \tag{27}$$

where the error term E_2' in (27) can be estimated as

$$E_2' \ll \frac{Q}{\delta_3}\frac{Q}{\delta_4}\left(\frac{(\delta_3+\delta_4)(T_3+T_4)}{Q}\right) \ll (T_3+T_4)Q. \tag{28}$$

3.3.3. **Third Step.** For fixed q_1, q_2, k_3, k_4, let K' be a large positive integer to be chosen later and let T' be a real number such that

$$T'K' = aq_1q_2 \leq Q^2.$$

We can now rewrite the inner sum of the main term of $\sum_2$ from (27) as

$$\sum\nolimits_3 = \sum_{1\leq k'\leq K'} \sum_{\substack{(k_3-1)T_3<q_3\leq k_3T_3,\\ (k_4-1)T_4<q_4\leq k_4T_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,\\ (k'-1)T'<\widetilde{a}_2\leq k'T',\\ \widetilde{a}_2\cdot bq_3q_4\equiv 1 \pmod{aq_1q_2}}} f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right). \tag{29}$$

For $(k'-1)T' < \widetilde{a}_2 \leq k'T'$, we have

$$\frac{(k'-1)T'}{aq_1q_2} < \frac{\widetilde{a}_2}{aq_1q_2} \leq \frac{k'T'}{aq_1q_2} = \frac{k'}{K'},$$

so that

$$\left|\frac{\widetilde{a}_2}{aq_1q_2} - \frac{k'}{K'}\right| \leq \frac{T'}{aq_1q_2} = \frac{1}{K'},$$

and

$$\left|f\left(\frac{\widetilde{a}_2}{aq_1q_2}\right) - f\left(\frac{k'}{K'}\right)\right| \leq ||Df||_\infty \cdot \left|\frac{\widetilde{a}_2}{aq_1q_2} - \frac{k'}{K'}\right| \ll \frac{1}{K'}.$$

Therefore (29) becomes

$$\sum\nolimits_3 = \sum_{1\leq k'\leq K'} f\left(\frac{k'}{K'}\right) \sum_{\substack{(k_3-1)T_3<q_3\leq k_3T_3,\\ (k_4-1)T_4<q_4\leq k_4T_4,\\ (q_3,q_1q_2\delta)=1,\\ (q_4,q_1q_2q_3\delta)=1,\\ (k'-1)T'<\widetilde{a}_2\leq k'T',\\ \widetilde{a}_2\cdot bq_3q_4\equiv 1 \pmod{aq_1q_2}}} 1 + E_3', \tag{30}$$

where the error term E_3' in (30) can be estimated as

$$E_3' \ll \frac{T_3T_4}{K'} = \frac{Q^2}{\delta_3\delta_4K'K_3K_4}. \tag{31}$$

3.4. **A Counting Lemma.** For fixed q_1, q_2, k_3, k_4, our next goal is to estimate the inner sum of the main term of $\sum_3$ from (30), which can be written in the simpler form

$$\sum\nolimits_4 = \sum_{\substack{m\in\mathbf{I}, n\in\mathbf{J},\\ (m,n)=(mn,\delta)=1,\\ \overline{bmn}/q \in(\alpha,\beta]}} 1,$$

with $\mathbf{I} = ((k_3-1)T_3, k_3T_3] \subset (0, Q/\delta_3], \mathbf{J} = ((k_4-1)T_4, k_4T_4] \subset (0, Q/\delta_4]$,

$$q = aq_1q_2 \leq Q^2,$$

$\alpha = \frac{(k'-1)T'}{q}, \beta = \frac{k'T'}{q}, m = q_3, n = q_4$ where b is a fixed integer satisfying $(b,q) = 1$. Here for an integer x such that $(x,q) = 1$, we denote by $\bar{x}$ the multiplicative inverse of x modulo q, i.e. $0 < \bar{x} < q$ and $\bar{x}x \equiv 1 \pmod{q}$.

3.4.1. **First Step.** Defining the set

$$V := \left\{ x_{m,n} = \frac{\overline{bmn}}{q} : m \in \mathbf{I}, n \in \mathbf{J}, (m,n) = (mn, q\delta) = 1 \right\},$$

we have

$$\sum\nolimits_4 = \# \left(V \bigcap (\alpha, \beta] \right). \tag{32}$$

We will obtain the formula

$$\sum\nolimits_4 = \# \left(V \bigcap (\alpha, \beta] \right) = \frac{6T_3T_4}{\pi^2 K'} \prod_{p | \delta a q_1 q_2} \left(1 - \frac{2}{p+1} \right) + E_4$$

$$= \frac{6}{\pi^2 \delta_3 \delta_4} \frac{Q^2}{K_3 K_4 K'} \prod_{p | \delta q_1 q_2} \left(1 - \frac{2}{p+1} \right) + E_4$$

where the error term E_4 is to be estimated later. To this end first note that

$$\begin{aligned}
\#V &= \sum_{\substack{m \in \mathbf{I}, n \in \mathbf{J}, \\ (m,n) = (mn, q\delta) = 1,}} 1 = \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{\substack{n \in \mathbf{J}, \\ (n, mq\delta) = 1,}} 1 = \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{n \in \mathbf{J}} \sum_{\substack{d | n, \\ d | mq\delta,}} \mu(d) \\
&= \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{d | mq\delta,} \mu(d) \sum_{\substack{d | n, \\ n \in \mathbf{J},}} 1 = \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{d | mq\delta,} \mu(d) \left(\frac{|\mathbf{J}|}{d} + O(1) \right) \\
&= |\mathbf{J}| \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{d | mq\delta,} \frac{\mu(d)}{d} + O_\eta(|\mathbf{I}| Q^\eta) = |\mathbf{J}| \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \frac{\varphi(mq\delta)}{mq\delta} + O_\eta(|\mathbf{I}| Q^\eta) \\
&= |\mathbf{J}| \frac{\varphi(q\delta)}{q\delta} \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \frac{\varphi(m)}{m} + O_\eta(|\mathbf{I}| Q^\eta).
\end{aligned}$$

We observe that

$$\begin{aligned}
\sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \frac{\varphi(m)}{m} &= \sum_{\substack{m \in \mathbf{I}, \\ (m, q\delta) = 1,}} \sum_{d | m} \frac{\mu(d)}{d} = \sum_{\substack{d \leq Q, \\ (d, q\delta) = 1,}} \frac{\mu(d)}{d} \sum_{\substack{d | m, m \in \mathbf{I}, \\ (m, q\delta) = 1,}} 1 \\
&= \sum_{\substack{d \leq Q, \\ (d, q\delta) = 1,}} \frac{\mu(d)}{d} \sum_{\substack{m' \in \frac{\mathbf{I}}{d}, \\ (m', q\delta) = 1,}} 1.
\end{aligned}$$

Recall the elementary result that if the function $f \in C^1(\mathbb{R})$ has compact support, $\mathbf{I}$ is a finite interval, A is a fixed positive integer and l an integer, then

$$\left| \sum_{\substack{l \in \mathbf{I} \\ (l,A)=1}} f(l) - \frac{\varphi(A)}{A} \int_{\mathbf{I}} f(x)\mathrm{d}x \right| \leq \sigma_0(A) \left(||Df||_\infty |\mathbf{I}| + 2||f||_\infty \right), \tag{33}$$

where the number of divisors function satisfies

$$\sigma_0(A) = \sum_{d|A} 1 \ll_\epsilon A^\epsilon$$

for every fixed $\epsilon > 0$. Using this result we have

$$\begin{aligned}
\sum_{\substack{m \in \mathbf{I}, \\ (m,q\delta)=1,}} \frac{\varphi(m)}{m} &= \sum_{\substack{d \leq Q, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d} \left(\frac{\varphi(q\delta)}{q\delta} \int_{\frac{\mathbf{I}}{d}} 1 \,\mathrm{d}t + O_\eta\left((q\delta)^\eta\right) \right) \\
&= \sum_{\substack{d \leq Q, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d} \left(\frac{\varphi(q\delta)}{q\delta} \frac{|\mathbf{I}|}{d} + O_\eta\left(Q^\eta\right) \right) \\
&= \frac{\varphi(q\delta)}{q\delta} |\mathbf{I}| \sum_{\substack{d \leq Q, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d^2} + O_\eta\left(Q^\eta\right).
\end{aligned}$$

Completing the convergent sum above gives

$$\begin{aligned}
\sum_{\substack{d \leq Q, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d^2} &= \sum_{\substack{d \geq 1, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d^2} - \sum_{\substack{d > Q, \\ (d,q\delta)=1,}} \frac{\mu(d)}{d^2} \\
&= \frac{\prod_p \left(1 - \frac{1}{p^2}\right)}{\prod_{p|q\delta} \left(1 - \frac{1}{p^2}\right)} + O\left(\frac{1}{Q}\right).
\end{aligned}$$

Since

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2},$$

and $|\mathbf{I}| < Q$, we obtain

$$\sum_{\substack{m \in \mathbf{I}, \\ (m,q\delta)=1,}} \frac{\varphi(m)}{m} = \frac{\varphi(q\delta)}{q\delta} |\mathbf{I}| \frac{6}{\pi^2} \frac{1}{\prod_{p|q\delta} \left(1 - \frac{1}{p^2}\right)} + O_\eta\left(Q^\eta\right).$$

Inserting this into the above expression for $\#V$ finally gives

$$\begin{aligned}\#V &= |\mathbf{J}|\frac{\varphi(q\delta)}{q\delta}\left(\frac{\varphi(q\delta)}{q\delta}|\mathbf{I}|\frac{6}{\pi^2}\frac{1}{\prod_{p|q\delta}\left(1-\frac{1}{p^2}\right)}+O_\eta\left(Q^\eta\right)\right)+O_\eta(|\mathbf{I}|Q^\eta)\\ &= |\mathbf{I}|\cdot|\mathbf{J}|\frac{6}{\pi^2}\frac{\prod_{p|q\delta}\left(1-\frac{1}{p}\right)^2}{\prod_{p|q\delta}\left(1-\frac{1}{p^2}\right)}+O_\eta(|\mathbf{J}|Q^\eta)+O_\eta(|\mathbf{I}|Q^\eta)\\ &= \frac{6|\mathbf{I}|\cdot|\mathbf{J}|}{\pi^2}\prod_{p|q\delta}\left(1-\frac{2}{p+1}\right)+O_\eta((|\mathbf{I}|+|\mathbf{J}|)\cdot Q^\eta).\end{aligned}$$

3.4.2. **Second Step.** By the Erdös-Turán inequality([31]),

$$\begin{aligned}\left|\#\left(V\bigcap(\alpha,\beta]\right)-(\beta-\alpha)\#V\right| &\ll \frac{\#V}{L}+\sum_{1\le K\le L}\frac{1}{K}\left|\sum_{x_{m,n}\in V}e(Kx_{m,n})\right|\\ &\ll \frac{|\mathbf{I}|\cdot|\mathbf{J}|}{L}+\sum_{1\le K\le L}\frac{S_K}{K}.\end{aligned}$$

Here L is a large real number to be chosen later and

$$S_K=\left|\sum_{x_{m,n}\in V}e(Kx_{m,n})\right|.$$

Define, for $1\le K\le L$,

$$S(\mathbf{I},\mathbf{J},\delta,q,l)=\sum_{\substack{m\in\mathbf{I},\\(m,q\delta)=1,}}\sum_{\substack{n\in\mathbf{J},\\(n,q\delta)=1,}}e\left(\frac{l\bar{m}\bar{n}}{q}\right),$$

and note that taking $x_{m,n}=\frac{\bar{b}\bar{m}\bar{n}}{q}$, S_K can be rewritten as

$$S_K = \left|\sum_{\substack{m\in\mathbf{I},\\(m,q\delta)=1,}}\sum_{\substack{n\in\mathbf{J},\\(n,mq\delta)=1,}}e\left(\frac{K\bar{b}\bar{m}\bar{n}}{q}\right)\right| = \left|\sum_{\substack{m\in\mathbf{I},\\(m,q\delta)=1,}}\sum_{\substack{n\in\mathbf{J},\\(n,q\delta)=1,}}e\left(\frac{K\bar{b}\bar{m}\bar{n}}{q}\right)\sum_{\substack{d|n,\\d|m}}\mu(d)\right| \tag{34}$$

(35)

$$= \left|\sum_{\substack{d\le Q,\\(d,q\delta)=1}}\mu(d)\sum_{\substack{m\in\frac{\mathbf{I}}{d},\\(m,q\delta)=1,}}\sum_{\substack{n\in\frac{\mathbf{J}}{d},\\(n,q\delta)=1,}}e\left(\frac{K\overline{bd^2}\bar{m}\bar{n}}{q}\right)\right|$$

$$= \left| \sum_{\substack{d \le Q, \\ (d,q\delta)=1}} \mu(d) S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right) \right|.$$

We use the trivial estimate

$$\left| S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right) \right| \le \left(\frac{|\mathbf{I}|}{d} + 1\right)\left(\frac{|\mathbf{J}|}{d} + 1\right)$$

$$= \frac{|\mathbf{I}| \cdot |\mathbf{J}|}{d^2} + \frac{|\mathbf{I}|}{d} + \frac{|\mathbf{J}|}{d} + 1,$$

and let R be a real number to be chosen later with $0 < R < |\mathbf{I}|$, to deduce that

$$\left| \sum_{\substack{R < d \le Q, \\ (d,q\delta)=1}} \mu(d) S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right) \right| \le \sum_{R < d \le Q,} \left(\frac{|\mathbf{I}| \cdot |\mathbf{J}|}{d^2} + \frac{|\mathbf{I}|}{d} + \frac{|\mathbf{J}|}{d} + 1\right) \tag{36}$$

$$\le |\mathbf{I}| \cdot |\mathbf{J}| \sum_{d > R} \frac{1}{d^2} + (|\mathbf{I}| + |\mathbf{J}|) \sum_{d > R} \frac{1}{d} + Q$$

$$\ll \frac{|\mathbf{I}| \cdot |\mathbf{J}|}{R} + (|\mathbf{I}| + |\mathbf{J}|) \log Q + Q.$$

The main difficulty comes from small values of d, namely if $1 \le d \le R$, then we write

$$S = S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right) = \sum_{\substack{u \in \frac{\mathbf{I}}{d}, \\ (u,q\delta)=1,}} \sum_{\substack{v \in \frac{\mathbf{J}}{d}, \\ (v,q\delta)=1,}} e\left(\frac{K\overline{bd^2}\bar{u}\bar{v}}{q}\right).$$

Applying Hölder's inequality and noting that $(u, q\delta) = 1$ implies $(u, q) = 1$, we have

$$|S|^4 \le \left(\sum_{\substack{u \in \frac{\mathbf{I}}{d}, \\ (u,q\delta)=1,}} 1\right)^3 \cdot \left(\sum_{\substack{u \in \frac{\mathbf{I}}{d}, \\ (u,q)=1,}} \left| \sum_{\substack{v \in \frac{\mathbf{J}}{d}, \\ (v,q\delta)=1,}} e\left(\frac{K\overline{bd^2}\bar{u}\bar{v}}{q}\right) \right|^4\right) \tag{37}$$

We will distinguish two cases, namely that $q \le \frac{|\mathbf{I}|}{d}$ and $q > \frac{|\mathbf{I}|}{d}$. First assuming $q \le \frac{|\mathbf{I}|}{d}$, observe that there are $\ll \frac{|\mathbf{I}|}{dq}$ consecutive intervals of length q covering all of the interval $\frac{\mathbf{I}}{d}$. Therefore we obtain from (36) that

$$|S|^4 \ll \left(\frac{|\mathbf{I}|}{d}\right)^4 \frac{1}{q} \sum_{\substack{v_1, v_2, v_3, v_4 \in \frac{\mathbf{J}}{d}, \\ (v_i, q\delta)=1,}} \sum_{\substack{1 \le u \le q, \\ (u,q)=1,}} e\left(\frac{K\overline{bd^2}\bar{u}(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4)}{q}\right). \tag{38}$$

As u runs through a reduced residue system modulo q, then so is $\overline{bd^2}\bar{u}$, so that (37) becomes

$$|S|^4 \ll \left(\frac{|\mathbf{I}|}{d}\right)^4 \frac{1}{q} \sum_{\substack{v_1,v_2,v_3,v_4 \in \frac{\mathbf{J}}{d},\\ (v_i,q\delta)=1,}} \sum_{\substack{1\le u\le q,\\ (u,q)=1,}} e\left(\frac{Ku(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4})}{q}\right) \tag{39}$$

$$= \left(\frac{|\mathbf{I}|}{d}\right)^4 \frac{1}{q} \sum_{\substack{v_1,v_2,v_3,v_4 \in \frac{\mathbf{J}}{d},\\ (v_i,q\delta)=1,}} \sum_{1\le u\le q} e\left(\frac{Ku(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4})}{q}\right) \sum_{\substack{s|u,\\ s|q,}} \mu(s)$$

$$= \left(\frac{|\mathbf{I}|}{d}\right)^4 \frac{1}{q} \sum_{s|q} \mu(s) \sum_{\substack{v_1,v_2,v_3,v_4 \in \frac{\mathbf{J}}{d},\\ (v_i,q\delta)=1,}} \sum_{1\le t\le \frac{q}{s}} e\left(\frac{Kt(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4})}{\frac{q}{s}}\right).$$

Using the fact that

$$\sum_{1\le t\le \frac{q}{s}} e\left(\frac{Kt(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4})}{\frac{q}{s}}\right) = \frac{q}{s}$$

when

$$K(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4}) \equiv 0 \pmod{\frac{q}{s}}$$

and zero otherwise, we have from (38) that

$$|S|^4 \ll \left(\frac{|\mathbf{I}|}{d}\right)^4 \sum_{s|q} \frac{1}{s} \times \tag{40}$$

$$\#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i,q\delta)=1 \\ K(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\}.$$

A similar argument for the case $q > \frac{|\mathbf{I}|}{d}$ gives

$$|S|^4 \ll \left(\frac{|\mathbf{I}|}{d}\right)^3 q \sum_{s|q} \frac{1}{s} \times \tag{41}$$

$$\#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i,q\delta)=1 \\ K(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\}.$$

Since $\frac{|\mathbf{I}|}{d} < Q$ and $q \le Q^2$, (39) and (40) can be combined under the single estimate

$$|S|^4 \ll \left(\frac{|\mathbf{I}|}{d}\right)^3 Q^2 \sum_{s|q} \frac{1}{s} \times \tag{42}$$

$$\#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i,q\delta)=1 \\ K(\bar{v_1}+\bar{v_2}-\bar{v_3}-\bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\}.$$

We need to control the number of all admissible tuples (v_1,v_2,v_3,v_4) appearing in (41). Although it is possible to obtain reasonable upper bounds for individual q, the quality of these bounds would not be good enough to arrive at an error term which is $o(Q^4)$. Therefore we prefer to average over all $q \le Q^2$. Clearly the condition

$$K(\bar{v_1} + \bar{v_2} - \bar{v_3} - \bar{v_4}) \equiv 0 \pmod{\frac{q}{s}}$$

for $s \mid q$ implies

$$K\left(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)\right) \equiv 0 \pmod{\frac{q}{s}}.$$

Consequently we have

$$\begin{aligned} &\sum_{q \le Q^2} \sum_{s \mid q} \frac{1}{s} \times \\ &\#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(\bar{v_1} + \bar{v_2} - \bar{v_3} - \bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} \\ &\le \sum_{q \le Q^2} \sum_{s \mid q} \frac{1}{s} \times \\ &\#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} \\ &= \sum_{s \le Q^2} \frac{1}{s} \sum_{\substack{q \le Q^2 \\ s \mid q}} \#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\}. \end{aligned} \tag{43}$$

Fixing $s \le Q^2$ temporarily, we observe that

$$\begin{aligned} &\sum_{\substack{q \le Q^2 \\ s \mid q}} \#\left\{(v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} \\ &\le \#\left\{(q,v_1,v_2,v_3,v_4) \,\middle|\, \begin{array}{c} q \le Q^2, \quad v_i \in \frac{\mathbf{J}}{d}, \quad s \mid q \\ K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\}. \end{aligned} \tag{44}$$

In order to find a useful upper bound for the number of admissible tuples (q, v_1, v_2, v_3, v_4), we have to distinguish two cases. First of all, if $v_1v_2(v_3 + v_4) \neq v_3v_4(v_1 + v_2)$, then using the fact that

$$\frac{q}{s} \mid K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2))$$

and

$$0 \neq |K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2))| \le 2Q^5,$$

it follows that the number of such integers $\frac{q}{s}$ is bounded by the number of divisors of $K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2))$, which is $\ll_\eta Q^\eta$ for $\eta > 0$. Since s is fixed, for each tuple (v_1, v_2, v_3, v_4), the number of admissible q is again $\ll_\eta Q^\eta$. In conclusion, the number of all admissible tuples (q, v_1, v_2, v_3, v_4) is

$$\ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^4 Q^\eta.$$

In the other case, if $v_1v_2(v_3+v_4) = v_3v_4(v_1+v_2)$, then fix $v_3, v_4 \in \frac{\mathbf{J}}{d} \cap \mathbb{Z}$ and put

$$\frac{1}{v_3} + \frac{1}{v_4} = \frac{a}{b},$$

where a, b are integers with $(a,b) = 1, |a|, |b| \le Q^2$. We consider solutions of the equation

$$\frac{1}{v_1} + \frac{1}{v_2} = \frac{a}{b}$$

for $v_1, v_2 \in \frac{\mathbf{J}}{d} \cap \mathbb{Z}$. Equivalently, we may write $b(v_1+v_2) = av_1v_2$. Taking $(v_1, v_2) = \widetilde{d}, v_1 = \widetilde{d}n, v_2 = \widetilde{d}m$ with $(m,n) = 1$, gives

$$b(m+n) = a\widetilde{d}mn.$$

Using $(m, m+n) = (n, m+n) = 1$, we have $m|b, n|b$, and the number of such pairs (m,n) is $\ll_\eta b^\eta \ll_\eta Q^\eta$. Since $\widetilde{d}|b(m+n)$, the number of $\widetilde{d}$ for fixed m, n is $\ll_\eta Q^\eta$. Therefore for fixed v_3, v_4, the number of such pairs (v_1, v_2) is also $\ll_\eta Q^\eta$. Observing that there are at most Q^2 choices for q and at most $\ll \left(\frac{|\mathbf{J}|}{d}\right)^2$ choices for the pairs (v_3, v_4), the number of admissible tuples (q, v_1, v_2, v_3, v_4) in this case is

$$\ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^2 Q^{2+\eta}.$$

Using $\frac{|\mathbf{J}|}{d} < Q$ and combining the two cases, we see that

$$\#\left\{(q, v_1, v_2, v_3, v_4) \,\middle|\, \begin{array}{c} q \le Q^2, \quad v_i \in \frac{\mathbf{J}}{d}, \quad s \mid q \\ K(v_1v_2(v_3+v_4) - v_3v_4(v_1+v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} \tag{45}$$

$$\ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^2 Q^{2+\eta}$$

for $\eta > 0$. Combining (42), (43) and (44), we deduce

$$\sum_{q \le Q^2} \sum_{s|q} \frac{1}{s} \#\left\{(v_1, v_2, v_3, v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(\bar{v_1} + \bar{v_2} - \bar{v_3} - \bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} \tag{46}$$

$$\ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^2 Q^{2+\eta} \sum_{s \le Q^2} \frac{1}{s} \ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^2 Q^{2+\eta} \log Q \ll_\eta \left(\frac{|\mathbf{J}|}{d}\right)^2 Q^{2+\eta}$$

for $\eta > 0$. Let $0 < \sigma < 1$ be a parameter which we will fix later. As a result of (45), the number of $q \le Q^2$ such that

$$\sum_{s|q} \frac{1}{s} \#\left\{(v_1, v_2, v_3, v_4) \,\middle|\, \begin{array}{c} v_i \in \frac{\mathbf{J}}{d}, \quad (v_i, q\delta) = 1 \\ K(\bar{v_1} + \bar{v_2} - \bar{v_3} - \bar{v_4}) \equiv 0 \pmod{\frac{q}{s}} \end{array}\right\} > \left(\frac{|\mathbf{J}|}{d}\right)^{3-\sigma} \tag{47}$$

is

$$\ll_\eta \frac{Q^{2+\eta}}{\left(\frac{|\mathbf{J}|}{d}\right)^{1-\sigma}}.$$

Let $B_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)$ be the set of all $q \leq Q^2$ such that (46) holds. Clearly we have

$$\left|B_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)\right| \ll_\eta \frac{Q^{2+\eta}}{\left(\frac{|\mathbf{J}|}{d}\right)^{1-\sigma}}. \tag{48}$$

Also let $G_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)$ be the complementary set of $B_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)$ in $(0,Q^2]$. If $q \leq Q^2$ and $q \in G_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)$, then it follows from (41) that

$$|S|^4 \leq \left(\frac{|\mathbf{I}|}{d}\right)^3 Q^2 \left(\frac{|\mathbf{J}|}{d}\right)^{3-\sigma},$$

and consequently that

$$|S| \leq Q^{\frac{1}{2}} \left(\frac{|\mathbf{I}|}{d}\right)^{\frac{3}{4}} \left(\frac{|\mathbf{J}|}{d}\right)^{\frac{3-\sigma}{4}} = Q^{\frac{1}{2}} \frac{|\mathbf{I}|^{\frac{3}{4}} \cdot |\mathbf{J}|^{\frac{3-\sigma}{4}}}{d^{\frac{6-\sigma}{4}}}. \tag{49}$$

Therefore if $q \in \bigcap_{d \leq R} G_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)$, then

$$\left|\sum_{\substack{d \leq R,\\ (d,q\delta)=1}} \mu(d) S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right)\right| \leq \sum_{d \leq R,} Q^{\frac{1}{2}} \frac{|\mathbf{I}|^{\frac{3}{4}} \cdot |\mathbf{J}|^{\frac{3-\sigma}{4}}}{d^{\frac{6-\sigma}{4}}} \tag{50}$$

$$\ll Q^{\frac{1}{2}} |\mathbf{I}|^{\frac{3}{4}} \cdot |\mathbf{J}|^{\frac{3-\sigma}{4}},$$

since

$$\frac{6-\sigma}{4} > \frac{5}{4} > 1.$$

In conclusion, for

$$q \in \bigcap_{1 \leq K \leq L} \bigcap_{d \leq R} G_{\sigma,K,\frac{\mathbf{J}}{d}}(Q),$$

one has from (34), (35) and (49) that

$$S_K \leq \left|\sum_{\substack{R<d \leq Q,\\ (d,q\delta)=1}} \mu(d) S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right)\right| + \left|\sum_{\substack{d \leq R,\\ (d,q\delta)=1}} \mu(d) S\left(\frac{\mathbf{I}}{d}, \frac{\mathbf{J}}{d}, \delta, q, K\overline{bd^2}\right)\right| \tag{51}$$

$$\ll \left(\frac{|\mathbf{I}| \cdot |\mathbf{J}|}{R} + (|\mathbf{I}| + |\mathbf{J}|) \log Q + Q\right) + \left(Q^{\frac{1}{2}} |\mathbf{I}|^{\frac{3}{4}} \cdot |\mathbf{J}|^{\frac{3-\sigma}{4}}\right).$$

Finally we obtain for such q that

$$\left|\#\left(V \bigcap (\alpha,\beta]\right) - (\beta-\alpha)\#V\right| \ll \frac{|\mathbf{I}| \cdot |\mathbf{J}|}{L} + \sum_{1 \leq K \leq L} \frac{S_K}{K} \tag{52}$$

$$\ll \frac{|\mathbf{I}| \cdot |\mathbf{J}|}{L} + \left(\frac{|\mathbf{I}| \cdot |\mathbf{J}|}{R} + (|\mathbf{I}| + |\mathbf{J}|) \log Q + Q + Q^{\frac{1}{2}} |\mathbf{I}|^{\frac{3}{4}} \cdot |\mathbf{J}|^{\frac{3-\sigma}{4}}\right) \log L.$$

3.4.3. **Third Step.** Recall that

$$|\mathbf{I}| = T_3, |\mathbf{J}| = T_4, q = aq_1q_2, \text{ and } \beta - \alpha = \frac{T'}{aq_1q_2} = \frac{1}{K'}.$$

Choosing

$$R = L < T_3 < Q,$$

we see that

$$\log R = \log L \le \log Q \ll_\eta Q^\eta.$$

In this way (51) becomes

$$\left|\# \left(V \bigcap (\alpha, \beta]\right) - (\beta - \alpha)\# V\right| \ll_\eta \left(\frac{T_3T_4}{L} + Q + Q^{\frac{1}{2}} T_3^{\frac{3}{4}} \cdot T_4^{\frac{3-\sigma}{4}}\right) Q^\eta.$$

Recall that

$$\# V = \frac{6T_3T_4}{\pi^2} \prod_{p|\delta q} \left(1 - \frac{2}{p+1}\right) + O_\eta((T_3 + T_4) \cdot Q^\eta)$$

and

$$(\beta - \alpha) \cdot \# V = \frac{6T_3T_4}{\pi^2 K'} \prod_{p|\delta q} \left(1 - \frac{2}{p+1}\right) + O_\eta\left(\frac{T_3 + T_4}{K'} \cdot Q^\eta\right).$$

Since $K_3T_3 = Q/\delta_3, K_4T_4 = Q/\delta_4$ and $a|\delta$, from (32) we obtain, as promised in the beginning of Section 3.4, that

$$\sum\nolimits_4 = \# \left(V \bigcap (\alpha, \beta]\right) = \frac{6T_3T_4}{\pi^2 K'} \prod_{p|\delta a q_1 q_2} \left(1 - \frac{2}{p+1}\right) + E_4 \tag{53}$$

$$= \frac{6}{\pi^2 \delta_3 \delta_4} \frac{Q^2}{K_3K_4K'} \prod_{p|\delta q_1 q_2} \left(1 - \frac{2}{p+1}\right) + E_4,$$

where the error term E_4 is estimated as

$$E_4 \ll_\eta \frac{T_3 + T_4}{K'} \cdot Q^\eta + \left(\frac{T_3T_4}{L} + Q + Q^{\frac{1}{2}} T_3^{\frac{3}{4}} \cdot T_4^{\frac{3-\sigma}{4}}\right) Q^\eta \tag{54}$$

$$\ll_\eta \left(\frac{T_3T_4}{L} + Q + Q^{\frac{1}{2}} T_3^{\frac{3}{4}} \cdot T_4^{\frac{3-\sigma}{4}}\right) Q^\eta.$$

3.5. **Estimation of Error Terms.**

3.5.1. **First Step.** Denoting

$$U_\sigma := \bigcap \left\{ G_{\sigma,K,\frac{\mathbf{J}}{d}}(Q) \;\middle|\; \begin{array}{ll} 1 \le d \le L, & 1 \le K \le L \\ 1 \le k_4 \le K_4, & \mathbf{J} = ((k_4-1)T_4, k_4T_4] \end{array} \right\},$$

for $aq_1q_2 \in U_\sigma$, and gathering $\sum_3$, $\sum_4$ from (30), (52) we have

$$\begin{aligned} \sum\nolimits_3 &= \sum_{1\le k'\le K'} f\left(\frac{k'}{K'}\right)\left(\frac{6Q^2}{\pi^2\delta_3\delta_4K_3K_4K'}\prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)+E_4\right)+E_3' \\ &= \frac{6Q^2}{\pi^2\delta_3\delta_4K_3K_4}\prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\frac{1}{K'}\sum_{1\le k'\le K'} f\left(\frac{k'}{K'}\right)+E_3'', \end{aligned}$$

where

$$E_3'' \ll K'E_4 + E_3',$$

and using the estimates for E_4 and E_3' from (31), (53) we get

$$E_3'' \ll_\eta \left(\frac{T_3T_4}{L} + Q + Q^{\frac{1}{2}}T_3^{\frac{3}{4}}\cdot T_4^{\frac{3-\sigma}{4}}\right)K'Q^\eta + \frac{Q^2}{\delta_3\delta_4K_3K_4K'}. \tag{55}$$

Recall that if the function $f \in C^1(\mathbb{R})$ and K is any positive integer, then

$$\left|\frac{1}{K}\sum_{k=1}^{K} f\left(\frac{k}{K}\right) - \int_0^1 f(x)\mathrm{d}x\right| \le \frac{||Df||_\infty}{K}. \tag{56}$$

Using this elementary result and (16), one has

$$\frac{1}{K'}\sum_{1\le k'\le K'} f\left(\frac{k'}{K'}\right) = \int_0^1 f(x)\mathrm{d}x + O\left(\frac{1}{K'}\right),$$

so that

$$\sum\nolimits_3 = \frac{6Q^2}{\pi^2\delta_3\delta_4K_3K_4}\prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\int_0^1 f(x)\mathrm{d}x + E_3,$$

where we can estimate E_3 as

$$E_3 \ll E_3'' + \frac{Q^2}{\delta_3\delta_4K_3K_4K'}. \tag{57}$$

3.5.2. **Second Step.** Going back to $\sum_2$ from (27) and using the error term, for $aq_1q_2 \in U_\sigma$ we have

$$\begin{aligned}\sum\nolimits_2 &= \sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} h\left(q_1, q_2, \frac{Q}{\delta_3}\frac{k_3}{K_3}, \frac{Q}{\delta_4}\frac{k_4}{K_4}\right) \times \\ &\quad \left(\frac{6Q^2}{\pi^2\delta_3\delta_4 K_3K_4}\prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\int_0^1 f(x)\mathrm{d}x + E_3\right) + E_2' \\ &= \frac{6Q^2}{\pi^2\delta_3\delta_4}\int_0^1 f(x)\mathrm{d}x \prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\times \\ &\quad \frac{1}{K_3K_4}\sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} h\left(q_1, q_2, \frac{Q}{\delta_3}\frac{k_3}{T_3}, \frac{Q}{\delta_4}\frac{k_4}{T_4}\right) + E_2'',\end{aligned}$$

where

$$E_2'' \ll K_3K_4E_3 + E_2' \ll K_3K_4E_3 + (T_3+T_4)Q. \tag{58}$$

Applying (56) two times to the sum

$$\frac{1}{K_3K_4}\sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} h\left(q_1, q_2, \frac{Q}{\delta_3}\frac{k_3}{T_3}, \frac{Q}{\delta_4}\frac{k_4}{T_4}\right),$$

we get

$$\begin{aligned}\frac{1}{K_3K_4}\sum_{\substack{1\le k_3\le K_3,\\ 1\le k_4\le K_4,}} h\left(q_1, q_2, \frac{Q}{\delta_3}\frac{k_3}{T_3}, \frac{Q}{\delta_4}\frac{k_4}{T_4}\right) &= \iint_{[0,1]^2} h\left(q_1, q_2, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w + \\ &\quad O\left(\frac{1}{K_3}+\frac{1}{K_4}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\sum\nolimits_2 &= \frac{6Q^2}{\pi^2\delta_3\delta_4}\left(\int_0^1 f(x)\mathrm{d}x\right)\prod_{p|\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\times \\ &\quad \iint_{[0,1]^2} h\left(q_1, q_2, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w + E_2,\end{aligned}$$

where

$$E_2 \ll Q^2\left(\frac{1}{K_3}+\frac{1}{K_4}\right) + E_2''. \tag{59}$$

3.5.3. **Third Step.** Now returning to $\sum_1$ from (24), we may write it as

$$\begin{aligned}\sum\nolimits_1 &= \sum_{\substack{q_1\le Q/\delta_1,\\ q_2\le Q/\delta_2,\\ (q_1,q_2\delta)=1,\\ (q_2,\delta)=1,}} \left(\sum_{aq_1q_2\in U_\sigma} + \sum_{aq_1q_2\notin U_\sigma}\right)\\ &= \sum\nolimits_1' + \sum\nolimits_1''.\end{aligned}$$

As we know the complement of $\overline{U}_\sigma$ is given by

$$\begin{aligned}\overline{U}_\sigma &:= \bigcup\left\{B_{\sigma,K,\frac{\mathbf{J}}{d}}(Q)\;\middle|\; \begin{matrix}1\le d\le L, & 1\le K\le L\\ 1\le k_4\le K_4, & \mathbf{J}=((k_4-1)T_4,k_4T_4]\end{matrix}\right\}\\ &\subset (0,Q^2],\end{aligned}$$

and using (47),

$$\#\overline{U}_\sigma \ll_\eta \sum_{\substack{1\le k_4\le K_4,\\ 1\le K\le L,\\ 1\le d\le L,}} \frac{Q^{2+\eta}}{\left|\frac{T_4}{d}\right|^{1-\sigma}} = K_4L\frac{Q^{2+\eta}}{T_4^{1-\sigma}}\sum_{d\le L} d^{1-\sigma} \ll K_4L^{3-\sigma}\cdot\frac{Q^{2+\eta}}{T_4^{1-\sigma}}.$$

Since the number of divisors of every integer in $\overline{U}_\sigma$ is $\ll_\eta Q^\eta$, the number of triples (a,q_1,q_2) with $aq_1q_2\in\overline{U}_\sigma$ fixed is $\ll_\eta Q^{3\eta}$ for $\eta>0$ and therefore

$$\#\{(a,q_1,q_2)\in\mathbb{N} : aq_1q_2\in\overline{U}_\sigma\} \ll_\eta Q^\eta\cdot\#\overline{U}_\sigma,$$

for every fixed $\eta>0$. It follows that

$$E_1''=\sum\nolimits_1'' \ll_\eta Q^\eta\cdot\#\overline{U}_\sigma\cdot\frac{Q^2}{\delta_1\delta_2} \ll_\eta K_4L^{3-\sigma}\cdot\frac{Q^{4+\eta}}{T_4^{1-\sigma}\delta_1\delta_2}. \tag{60}$$

Combining this with the result of the Second Step, for fixed a, we have

$$\begin{aligned}\sum\nolimits_1' &= \sum_{\substack{q_1\le Q/\delta_1,\\ q_2\le Q/\delta_2,\\ (q_1,q_2\delta)=1,\\ (q_2,\delta)=1,}} \sum_{aq_1q_2\in U_\sigma}\\ &= \sum_{\substack{q_1\le Q/\delta_1,\\ q_2\le Q/\delta_2,\\ (q_1,q_2\delta)=1,\\ (q_2,\delta)=1,\\ aq_1q_2\in U_\sigma}} \Big[\frac{6Q^2}{\pi^2\delta_3\delta_4}\left(\int_0^1 f(x)\mathrm{d}x\right)\prod_{p\mid\delta q_1q_2}\left(1-\frac{2}{p+1}\right)\times\\ &\qquad \iint_{[0,1]^2} h\left(q_1,q_2,\frac{Q}{\delta_3}z,\frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w + E_2\Big].\end{aligned}$$

Therefore one has

$$\sum\nolimits_1 = \sum_{\substack{q_1 \le Q/\delta_1,\\ q_2 \le Q/\delta_2,\\ (q_1, q_2\delta)=1,\\ (q_2,\delta)=1,}} \Big[\frac{6Q^2}{\pi^2\delta_3\delta_4}\left(\int_0^1 f(x)\mathrm{d}x\right)\prod_{p|\delta q_1 q_2}\left(1-\frac{2}{p+1}\right)\times$$

$$\iint_{[0,1]^2} h\left(q_1, q_2, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w + E_2\Big] + O\left(E_1''\right)$$

$$= E_1 + \frac{6Q^2}{\pi^2\delta_3\delta_4}\left(\int_0^1 f(x)\mathrm{d}x\right)\prod_{p|\delta}\left(1-\frac{2}{p+1}\right)\times$$

$$\iint_{[0,1]^2} \sum_{\substack{q_1 \le Q/\delta_1,\\ q_2 \le Q/\delta_2,\\ (q_1, q_2\delta)=1,\\ (q_2,\delta)=1,}} \prod_{p|q_1 q_2}\left(1-\frac{2}{p+1}\right) h\left(q_1, q_2, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w,$$

where

$$E_1 \ll E_1'' + Q^2 E_2. \tag{61}$$

In conclusion we have

$$\sum\nolimits_1 = E_1 + \frac{6Q^2}{\pi^2\delta_3\delta_4}\left(\int_0^1 f(x)\mathrm{d}x\right)\times \tag{62}$$

$$\iint_{[0,1]^2} \sum_{\substack{q_1 \le Q/\delta_1,\\ q_2 \le Q/\delta_2,\\ (q_1, q_2\delta)=1,\\ (q_2,\delta)=1,}} \prod_{p|\delta q_1 q_2}\left(1-\frac{2}{p+1}\right) h\left(q_1, q_2, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}z\,\mathrm{d}w,$$

where E_1 is estimated above in (61).

3.5.4. **Fourth Step.** Let us now complete the estimation of the error term E_1. Note that using $T_4K_4 = \frac{Q}{\delta_4}$ in (59), we have

$$E_1'' \ll_\eta \frac{K_4 L^{3-\sigma} Q^{4+\eta}}{T_4^{1-\sigma}\delta_1\delta_2} \ll_\eta K_4^{2-\sigma} L^{3-\sigma} Q^{3+\sigma+\eta}.$$

Also from (58),

$$E_2 \ll \frac{Q^2}{K_3} + \frac{Q^2}{K_4} + E_2'',$$

where by (57)

$$E_2'' \ll K_3K_4E_3 + (T_3+T_4)Q.$$

Finally combining (54) and (56) one has

$$E_3 \ll_\eta \left(\frac{T_3T_4}{L} + Q + Q^{\frac{1}{2}} T_3^{\frac{3}{4}} T_4^{\frac{3-\sigma}{4}}\right) K' Q^\eta + \frac{Q^2}{\delta_3\delta_4 K_3 K_4 K'}.$$

Gathering all of the above estimates together,

$$\begin{aligned} E_1 &\ll E_1'' + Q^2 E_2 \\ &\ll_\eta K_4^{2-\sigma} L^{3-\sigma} Q^{3+\sigma+\eta} + \frac{Q^4}{K_3} + \frac{Q^4}{K_4} + \frac{Q^4}{K'} + \\ &\quad Q^\eta \cdot \left(Q^4 \frac{K'}{L} + K_3 K_4 K' Q^3 + K' K_3^{\frac{1}{4}} K_4^{\frac{1+\sigma}{4}} Q^{4-\frac{\sigma}{4}} \right). \end{aligned}$$

Choosing

$$L = Q^{\sigma_1}, \quad K_3 = K_4 = K' \approx Q^{\sigma_2},$$

one obtains

$$\begin{aligned} E_1 &\ll_\eta Q^{(2-\sigma)\sigma_2 + (3-\sigma)\sigma_1 + 3 + \sigma + \eta} + Q^{4-\sigma_2} + \\ &\quad Q^\eta \cdot \left(Q^{4+\sigma_2-\sigma_1} + Q^{3+3\sigma_2} + Q^{\frac{6+\sigma}{4}\sigma_2 + 4 - \frac{\sigma}{4}} \right). \end{aligned}$$

Taking $\sigma_1 = 2\sigma_2$, this reduces to

$$E_1 \ll_\eta Q^\eta \left(Q^{3+\sigma(1-3\sigma_2)+8\sigma_2} + Q^{4-\sigma_2} + Q^{3+3\sigma_2} + Q^{\frac{6+\sigma}{4}\sigma_2 + 4 - \frac{\sigma}{4}} \right).$$

In order to balance all the terms above, it suffices to solve the system

$$\begin{cases} 3 + \sigma(1-3\sigma_2) + 8\sigma_2 = 4 - \sigma_2, \\ \frac{6+\sigma}{4}\sigma_2 + 4 - \frac{\sigma}{4} = 4 - \sigma_2, \end{cases}$$

with $0 < \sigma < 1$ to obtain that

$$\sigma_2 = \frac{10 - \sqrt{61}}{39} \approx \frac{1}{17.8} > \frac{1}{18}.$$

For convenience we may take $\sigma_2 = \frac{1}{18}, \sigma_1 = \frac{1}{9}, \sigma = \frac{46}{77}$ and arrive at the concluding estimate for E_1 as

$$E_1 \ll_\eta Q^{4-\frac{1}{18}+\eta}. \tag{63}$$

3.6. **Final Reductions.** Let

$$f_1(n) := \prod_{p|n} \left(1 - \frac{2}{p+1} \right),$$

and note that $f_1(n)$ is a multiplicative function with $|f_1(n)| \leq 1$. Our next objective is to eliminate the dependence on q_1, q_2 in the main term of $\sum_1$ from (61). To this end, we consider the sum over q_1, q_2 appearing in $\sum_1$ and put

$$\begin{aligned} W &= \sum_{\substack{q_1 \le Q/\delta_1, \\ q_2 \le Q/\delta_2, \\ (q_1, q_2\delta)=1, \\ (q_2,\delta)=1,}} f_1(\delta q_1 q_2) h\left(q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \\ &= \sum_{\substack{1 \le q_1 \le Q/\delta_1, \\ (q_1,\delta)=1,}} f_1(\delta q_1) \sum_{\substack{q_2 \le Q/\delta_2, \\ (q_2, q_1\delta)=1,}} f_1(q_2) h\left(q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right). \end{aligned}$$

If we define the Dirichlet convolution of μ and f_1 as

$$g_1(m) = (\mu * f_1)(m) = \sum_{d|m} \mu(d) f_1\left(\frac{m}{d}\right),$$

then $f_1 = 1 * g_1$ so that for any prime p,

$$g_1(p^n) = \begin{cases} -\frac{2}{p+1} & : \quad n = 1; \\ 0 & : \quad n \ge 2. \end{cases}$$

Therefore one has, $|g_1(p)| < \frac{2}{p}$, and

$$|g_1(d)| \le \frac{2^{\omega(d)}}{d} \text{ for every } d \ge 1,$$

where $\omega(d)$ is the number of distinct prime divisors of d. In this way the inner sum of W becomes

$$\begin{aligned} W_{q_1} &= \sum_{\substack{q_2 \le Q/\delta_2, \\ (q_2, q_1\delta)=1,}} f_1(q_2) h\left(q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \\ &= \sum_{\substack{q_2 \le Q/\delta_2, \\ (q_2, q_1\delta)=1,}} h\left(q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \sum_{D|q_2} g_1(D) \\ &= \sum_{\substack{D \le Q/\delta_2, \\ (D, q_1\delta)=1,}} g_1(D) \sum_{\substack{m \le \frac{Q}{D\delta_2}, \\ (m, q_1\delta)=1,}} h\left(q_1, mD, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right). \end{aligned}$$

For fixed q_1, z, w and D, define $F(m) := h(q_1, mD, z, w)$, and note that $||F||_\infty \ll 1$, and from (21),

$$|F'(m)| = D \cdot \left|\frac{\partial h}{\partial y}\right| \ll \frac{D\delta_2}{Q}.$$

Applying the result (33),

$$\sum_{\substack{m\le \frac{Q}{D\delta_2},\\ (m,q_1\delta)=1,}} h\left(q_1, mD, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right) = \frac{\varphi(\delta q_1)}{\delta q_1}\int_0^{\frac{Q}{D\delta_2}} h\left(q_1, Dy, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right) \mathrm{d}\, y + E_F$$

$$= \frac{\varphi(\delta q_1)}{\delta q_1}\frac{Q}{D\delta_2}\int_0^1 h\left(q_1, \frac{Q}{\delta_2}y, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right) \mathrm{d}\, y + E_F,$$

where the error term E_F is estimated as

$$E_F \ll_\eta (\delta q_1)^\eta \left(\frac{D\delta_2}{Q}\cdot\frac{Q}{D\delta_2}+1\right) \ll_\eta Q^\eta.$$

Consequently,

$$W_{q_1} = \sum_{\substack{D\le Q/\delta_2,\\ (D,q_1\delta)=1,}} g_1(D)\left(\frac{\varphi(\delta q_1)}{\delta q_1}\frac{Q}{\delta_2 D}\int_0^1 h\left(q_1, \frac{Q}{\delta_2}y, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right) \mathrm{d}\, y + E_F\right)$$

$$= \frac{Q}{\delta_2}\frac{\varphi(\delta q_1)}{\delta q_1}\left(\int_0^1 h\left(q_1, \frac{Q}{\delta_2}y, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right) \mathrm{d}\, y\right)\left(\sum_{\substack{D\le Q/\delta_2,\\ (D,q_1\delta)=1,}} \frac{g_1(D)}{D}\right) + E'_{q_1},$$

where the error term E'_{q_1} is estimated as

$$E'_{q_1} \ll \sum_{\substack{D\le Q/\delta_2,\\ (D,q_1\delta)=1,}} |g_1(D)|\cdot E_F \ll_\eta (\log Q)^2\cdot Q^\eta \ll_\eta Q^\eta,$$

since $g_1(D)\neq 0$ only for square-free D, $2^{\omega(D)}=\sigma_0(D)$ for such D and

$$\sum_{D\le Q/\delta_2} |g_1(D)| \le \sum_{D\le Q/\delta_2} \frac{\sigma_0(D)}{D} \ll (\log Q)^2$$

by partial summation. Completing the convergent sum on D, we have

$$\sum_{\substack{D\le Q/\delta_2,\\ (D,q_1\delta)=1,}} \frac{g_1(D)}{D} = \sum_{\substack{D\ge 1,\\ (D,q_1\delta)=1,}} \frac{g_1(D)}{D} + O\left(\sum_{D>Q/\delta_2}\frac{\mu^2(D)\sigma_0(D)}{D^2}\right)$$

$$= \frac{\prod_p\left(1+\frac{g_1(p)}{p}\right)}{\prod_{p|\delta q_1}\left(1+\frac{g_1(p)}{p}\right)} + O\left(\frac{\delta_2}{Q^{1-\eta}}\right)$$

$$= \frac{\prod_p\left(1-\frac{2}{p(p+1)}\right)}{\prod_{p|\delta q_1}\left(1-\frac{2}{p(p+1)}\right)} + O\left(\frac{\delta_2}{Q^{1-\eta}}\right).$$

Using this we may rewrite W_{q_1} as

$$W_{q_1} = \frac{Q}{\delta_2}\frac{\varphi(\delta q_1)}{\delta q_1}\frac{\prod_p\left(1-\frac{2}{p(p+1)}\right)}{\prod_{p|\delta q_1}\left(1-\frac{2}{p(p+1)}\right)}\left(\int_0^1 h\left(q_1, \frac{Q}{\delta_2}y, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}\,y\right) + E_{q_1}$$

where the error term E_{q_1} is still estimated as

$$E_{q_1} \ll_\eta Q^\eta.$$

Recall that

$$W = \sum_{\substack{1\le q_1\le Q/\delta_1,\\ (q_1,\delta)=1,}} f_1(\delta q_1)W_{q_1}.$$

Let

$$f_2(n) := \prod_{p|n}\left(1-\frac{3}{p+2}\right),\ n\in\mathbb{N}$$

and note that f_2 is a multiplicative function with $|f_2(n)|\le 1$. It follows that

$$\begin{aligned} f_1(n)\frac{\varphi(n)}{n}\frac{1}{\prod_{p|n}\left(1-\frac{2}{p(p+1)}\right)} &= \prod_{p|n}\frac{\left(1-\frac{2}{p+1}\right)\left(1-\frac{1}{p}\right)}{\left(1-\frac{2}{p(p+1)}\right)} \\ &= \prod_{p|n}\left(1-\frac{3}{p+2}\right) = f_2(n), \end{aligned}$$

and rewriting W gives

$$W = \frac{c_1 Q}{\delta_2}\int_0^1 \sum_{\substack{1\le q_1\le Q/\delta_1,\\ (q_1,\delta)=1,}} f_2(\delta q_1)h\left(q_1, \frac{Q}{\delta_2}y, \frac{Q}{\delta_3}z, \frac{Q}{\delta_4}w\right)\mathrm{d}\,y + E'_W,$$

where

$$c_1 = \prod_{p\text{ prime}}\left(1-\frac{2}{p(p+1)}\right) > 0,$$

and the error term can be estimated as

$$E'_W \ll \frac{Q}{\delta_1}\cdot E_{q_1} \ll_\eta Q^{1+\eta}.$$

Similarly, if we define $g_2 := \mu * f_2$, then $f_2 = 1 * g_2$ and for any prime p,

$$g_2(p^n) = \begin{cases} -\frac{3}{p+2} & : \ n=1; \\ 0 & : \ n\ge 2, \end{cases}$$

so that for every $d\ge 1$,

$$|g_2(d)| \le \frac{3^{\omega(d)}}{d} \le \frac{2^{2\omega(d)}}{d}.$$

We may now rewrite the inner sum inside the integral for W as

$$\begin{aligned} W_1 &= \sum_{\substack{q_1 \le Q/\delta_1, \\ (q_1,\delta)=1,}} f_2(\delta) f_2(q_1) h\left(q_1, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \\ &= f_2(\delta) \sum_{\substack{q_1 \le Q/\delta_1, \\ (q_1,\delta)=1,}} h\left(q_1, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \sum_{D \mid q_1} g_2(D) \\ &= f_2(\delta) \sum_{\substack{D \le Q/\delta_1, \\ (D,\delta)=1,}} g_2(D) \sum_{\substack{m \le \frac{Q}{D\delta_1}, \\ (m,\delta)=1,}} h\left(mD, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right). \end{aligned}$$

Using (33) and (20), we obtain

$$\begin{aligned} \sum_{\substack{m \le \frac{Q}{D\delta_1}, \\ (m,\delta)=1,}} h\left(mD, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) &= \frac{\varphi(\delta)}{\delta} \frac{Q}{D\delta_1} \times \\ & \int_0^1 h\left(\frac{Q}{\delta_1} x, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w\right) \mathrm{d}\, y + O(1), \end{aligned}$$

and completing the convergent sum on D,

$$\begin{aligned} \sum_{\substack{D \le Q/\delta_1, \\ (D,\delta)=1,}} \frac{g_2(D)}{D} &= \sum_{\substack{D \ge 1, \\ (D,\delta)=1,}} \frac{g_2(D)}{D} + O\left(\sum_{D > Q/\delta_1} \frac{\mu^2(D)\sigma_0^2(D)}{D^2}\right) \\ &= \frac{\prod_p \left(1 + \frac{g_2(p)}{p}\right)}{\prod_{p \mid \delta} \left(1 + \frac{g_2(p)}{p}\right)} + O\left(\frac{\delta_1}{Q^{1-\eta}}\right) \\ &= \frac{\prod_p \left(1 - \frac{3}{p(p+2)}\right)}{\prod_{p \mid \delta} \left(1 - \frac{3}{p(p+2)}\right)} + O\left(\frac{\delta_1}{Q^{1-\eta}}\right). \end{aligned}$$

Moreover using

$$|g_2(D)| \le \frac{\sigma_0^2(D)}{D}$$

and the well-known estimate

$$\sum_{m \le t} \sigma_0^2(m) \ll t(\log t)^3,$$

we obtain by partial summation that

$$\sum_{\substack{D \le Q/\delta_1, \\ (D,\delta)=1,}} |g_2(D)| \ll (\log Q)^4 \ll_\eta Q^\eta.$$

In conclusion we have

$$W_1 = f_2(\delta)\frac{Q}{\delta_1}\frac{\varphi(\delta)}{\delta}\frac{\prod_p\left(1-\frac{3}{p(p+2)}\right)}{\prod_{p|\delta}\left(1-\frac{3}{p(p+2)}\right)}\int_0^1 h\left(\frac{Q}{\delta_1}x,\frac{Q}{\delta_2}y,\frac{Q}{\delta_3}z,\frac{Q}{\delta_4}w\right)\mathrm{d}\,x+E^*,$$

where the error term E^* is estimated as

$$E^*\ll_\eta Q^\eta.$$

This completes the estimation of all error terms. Denoting

$$c_2=\prod_{p\text{ prime}}\left(1-\frac{3}{p(p+2)}\right)>0,$$

and the multiplicative function

$$c(n)=f_2(n)\frac{\varphi(n)}{n}\frac{1}{\prod_{p|n}\left(1-\frac{3}{p(p+2)}\right)}=\prod_{p|n}\left(1-\frac{4}{p+3}\right), \tag{64}$$

we rewrite $\sum_1$ from (61) as

$$\begin{aligned}\sum\nolimits_1 &= \frac{6c_1c_2c(\delta)Q^4}{\pi^2\delta_1\delta_2\delta_3\delta_4}\cdot\left(\int_0^1 f(x)\mathrm{d}\,x\right)\times\\ &\iint_{[0,1]^4} h\left(\frac{Q}{\delta_1}x,\frac{Q}{\delta_2}y,\frac{Q}{\delta_3}z,\frac{Q}{\delta_4}w\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w+E.\end{aligned}$$

Since $E_1\ll_\eta Q^{4-\frac{1}{18}+\eta}$, $Q^2E'_W\ll_\eta Q^{3+\eta}$ and $Q^3E^*\ll_\eta Q^{3+\eta}$, the final error term E can be estimated as

$$E\ll E_1+Q^2E'_W+Q^3E^*\ll_\eta Q^{4-\frac{1}{18}+\eta}+Q^{3+\eta}\ll_\eta Q^{4-\frac{1}{18}+\eta}, \tag{65}$$

for $0<\eta<\frac{1}{18}$.
Using (18) we have

$$h\left(\frac{Q}{\delta_1}x,\frac{Q}{\delta_2}y,\frac{Q}{\delta_3}z,\frac{Q}{\delta_4}w\right)\neq 0 \quad\Longrightarrow\quad c_\wedge\le x,y,z,w\le 1,$$

and combining (11),(12) and (14),

$$h\left(\frac{Q}{\delta_1}x,\frac{Q}{\delta_2}y,\frac{Q}{\delta_3}z,\frac{Q}{\delta_4}w\right)=H\left(\frac{Nm\delta r_1r_2r_3r_4}{Q^4xyzw}\right).$$

By (2), we have

$$\frac{N}{Q^4}=\frac{9}{\pi^4}+O_\eta\left(Q^{-1+\eta}\right),$$

so that

$$\left|H\left(\frac{Nm\delta r_1r_2r_3r_4}{Q^4xyzw}\right)-H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4xyzw}\right)\right|\ll_\eta Q^{-1+\eta}.$$

Letting $c = c_1c_2$ and rewriting $\sum_1$, we get

$$\begin{aligned}\sum\nolimits_1 &= \frac{6cc(\delta)Q^4}{\pi^2\delta_1\delta_2\delta_3\delta_4}\cdot\left(\int_0^1 f(x)\mathrm{d}\,x\right)\times \\ &\iint_{[0,1]^4} H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4 xyzw}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w + E,\end{aligned}$$

where

$$E \ll_\eta Q^{4-\frac{1}{18}+\eta}.$$

Recall that, by (11),

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x &= \int_0^1 G\left(\frac{mx+n}{\widetilde{u}}\right)^2\mathrm{d}x = \frac{\widetilde{u}}{m}\int_{\frac{n}{\widetilde{u}}}^{\frac{m+n}{\widetilde{u}}} G(z)^2\mathrm{d}z \\ &= \frac{\widetilde{u}}{m}\left(\int_{\frac{n}{\widetilde{u}}}^{\frac{n+1}{\widetilde{u}}} G(z)^2\mathrm{d}z + \cdots + \int_{\frac{n+m-1}{\widetilde{u}}}^{\frac{m+n}{\widetilde{u}}} G(z)^2\mathrm{d}z\right),\end{aligned}$$

and

$$\sum_{n\in\mathbb{Z}}\int_0^1 f(x)\mathrm{d}x = \frac{\widetilde{u}}{m}m\cdot\int_0^1 G(z)^2\mathrm{d}x = \widetilde{u}\int_0^1 G(z)^2\mathrm{d}x.$$

Recall that $\mathcal{A}$ is the finite set of all integers n satisfying

$$0 < \frac{\widetilde{a}_2 m}{[d_1,d_2]} + \frac{n}{\widetilde{u}} < 1.$$

Summing over all $n \in \mathcal{A}$, we obtain from (8) that

(66)
$$\sum_{\substack{a_i\le Q/r_i, 1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,}} \frac{1}{([a_1,a_2],[a_3,a_4])}\sum_{n\in\mathcal{A}}\sum\nolimits_1$$

$$= \sum_{\substack{a_i\le Q/r_i, 1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,\\ ([a_1,a_2],[a_3,a_4])=\widetilde{u},}} \frac{1}{\widetilde{u}}\left\{\frac{6cc(\delta)Q^4}{\pi^2\delta_1\delta_2\delta_3\delta_4}\widetilde{u}\left(\int_0^1 G(z)^2\mathrm{d}z\right)\times\iint_{[0,1]^4} H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4xyzw}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w + E\right\}$$

where $\sum_1$ is defined as in (9). If $\delta = 1$, then $P_\delta = \{1\}$ and $a_1 = a_2 = a_3 = a_4 = 1$. Otherwise $\delta \ge 2$, and we consider the prime factorization of δ as

(67)
$$\delta = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}, \quad e_1,\ldots,e_k \ge 1.$$

Writing the prime factorizations of a_i as

(68)
$$a_i = p_1^{e_{i1}}p_2^{e_{i2}}\cdots p_k^{e_{ik}}, \quad 1 \le i \le 4,$$

the condition $\frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta$ implies that

$$e_{1j}+e_{2j}+e_{3j}+e_{4j}-\max\{e_{1j},e_{2j},e_{3j},e_{4j}\}=e_j, \tag{69}$$

for $1\le j\le k$. Since $a_i\le Q/r_i\le Q$, it follows that

$$e_{ij}\log 2\le e_{i1}\log p_1+\cdots e_{ik}\log p_k\le \log Q-\log r_i,$$

and

$$0\le e_{ij}\le \frac{\log Q}{\log 2},$$

for any $1\le i\le 4$ and $1\le j\le k$. Since δ is absolutely bounded and $\omega(\delta)=k$, k is absolutely bounded and

$$\sum_{\substack{a_i\le Q/r_i,1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,}} 1\le \left(\frac{\log Q}{\log 2}\right)^{4k}\ll(\log Q)^{4k}\ll_\eta Q^\eta.$$

Since $\delta_i=a_ir_i$, $1\le i\le 4$, we also have from (65)

$$\sum_{\substack{a_i\le Q/r_i,1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,}} \frac{6cc(\delta)Q^4}{\pi^2\delta_1\delta_2\delta_3\delta_4}\left(\int_0^1 G(z)^2\mathrm{d}z\right)\times\iint_{[0,1]^4} H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4xyzw}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w+E \tag{70}$$

$$=\frac{6cc(\delta)}{\pi^2}\left(\int_0^1 G(z)^2\mathrm{d}z\right)\iint_{[0,1]^4} H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4xyzw}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\times$$

$$\frac{Q^4}{r_1r_2r_3r_4}\sum_{\substack{a_i\le Q/r_i,1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,}} \frac{1}{a_1a_2a_3a_4}+E\,,$$

where, by (64)

$$E\ll_\eta Q^{4-\frac{1}{18}+\eta}.$$

Our next goal is to estimate the sum

$$A_{\delta,r_1,r_2,r_3,r_4}(Q)=\sum_{\substack{a_i\le Q/r_i,1\le i\le 4,\\ a_1,a_2,a_3,a_4\in P_\delta,\\ \frac{a_1a_2a_3a_4}{[a_1,a_2,a_3,a_4]}=\delta,}} \frac{1}{a_1a_2a_3a_4}.$$

Again we may assume $\delta \geq 2$ and (67), (68), (69). Since $A_{\delta,r_1,r_2,r_3,r_4}(Q)$ is increasing as $Q \to \infty$, we deduce that

$$\begin{aligned} A_{\delta,r_1,r_2,r_3,r_4}(Q) &\leq \sum_{e_{ij}} \frac{1}{p_1^{e_{11}+e_{21}+e_{31}+e_{41}} p_2^{e_{12}+e_{22}+e_{32}+e_{42}} \cdots p_k^{e_{1k}+e_{2k}+e_{3k}+e_{4k}}} \\ &= \left(\sum_{e=0}^{\infty} \frac{1}{p_1^e}\right)^4 \cdots \left(\sum_{e=0}^{\infty} \frac{1}{p_k^e}\right)^4 \\ &= \left(\prod_{j=1}^{k} \frac{1}{1-p_j^{-1}}\right)^4 < \infty, \end{aligned}$$

and consequently

$$\lim_{Q\to\infty} A_{\delta,r_1,r_2,r_3,r_4}(Q) = A(\delta)$$

for some constant $A(\delta)$ depending only on δ. Denoting the condition (69) as $(e_{ij}) \in PP$, we obtain

$$A(p^m) = \sum_{\substack{e_{ij}\geq 0,\\ (e_{ij})\in PP}} \frac{1}{p_1^{e_{11}+e_{21}+e_{31}+e_{41}} p_2^{e_{12}+e_{22}+e_{32}+e_{42}} \cdots p_k^{e_{1k}+e_{2k}+e_{3k}+e_{4k}}}.$$

It is easy to see that A is a multiplicative function and for any prime number p and integer $m \geq 1$,

$$A(p^m) = \sum_{\substack{e_1,e_2,e_3,e_4\geq 0\\ e_1+e_2+e_3+e_4-\max\{e_1+e_2+e_3+e_4\}=m}} \frac{1}{p^{e_1+e_2+e_3+e_4}}\,. \tag{71}$$

Furthermore,

$$\begin{aligned} |A_{\delta,r_1,r_2,r_3,r_4}(Q) - A(\delta)| &< \sum_{e_{11}\log p_1+\cdots+e_{1k}\log p_k>\log Q-\log r_1} \cdots + \\ &\quad \sum_{e_{21}\log p_1+\cdots+e_{2k}\log p_k>\log Q-\log r_2} \cdots + \\ &\quad \sum_{e_{31}\log p_1+\cdots+e_{3k}\log p_k>\log Q-\log r_3} \cdots + \\ &\quad \sum_{e_{41}\log p_1+\cdots+e_{4k}\log p_k>\log Q-\log r_4} \cdots \\ &= \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4, \end{aligned}$$

where

$$\Omega_1 = \left(\prod_{j=1}^{k} \frac{1}{1-p_j^{-1}}\right)^3 \Omega_1',$$

and

$$\Omega_1' = \sum_{e_{11}\log p_1+\cdots+e_{1k}\log p_k>\log Q-\log r_1} \frac{1}{p_1^{e_{11}}\cdots p_k^{e_{1k}}}.$$

We can estimate Ω_1' as

$$\begin{aligned}\Omega_1' &< \sum_{e_{11}\log p_1>\log Q-\log r_1} \frac{1}{p_1^{e_{11}}\cdots p_k^{e_{1k}}} + \cdots + \sum_{e_{1k}\log p_k>\log Q-\log r_1} \frac{1}{p_1^{e_{11}}\cdots p_k^{e_{1k}}} \\ &\quad + \sum_{\substack{e_{11}\log p_1\le\log Q-\log r_1,\\ \vdots \\ e_{1k}\log p_k\le\log Q-\log r_1,\\ e_{11}\log p_1+\cdots+e_{1k}\log p_k>\log Q-\log r_1,}} \frac{1}{p_1^{e_{11}}\cdots p_k^{e_{1k}}} \\ &= \Omega_{1,1}+\cdots+\Omega_{1,k}+\Omega_1''.\end{aligned}$$

Note that for $\Omega_{1,1}$, $e_{12},\ldots,e_{1k}$ run through all nonnegative integers and since

$$\frac{1}{p_1^{e_{11}}} < \frac{r_1}{Q},$$

it follows that

$$\begin{aligned}\Omega_{1,1} &= \left(\prod_{j=2}^{k}\frac{1}{1-p_j^{-1}}\right)\sum_{e_{11}\log p_1>\log Q-\log r_1}\frac{1}{p_1^{e_{11}}} \\ &\le \left(\prod_{j=2}^{k}\frac{1}{1-p_j^{-1}}\right)\frac{\frac{r_1}{Q}}{1-p_1^{-1}} \ll \frac{1}{Q}.\end{aligned}$$

Similarly,

$$\Omega_{1,j} \ll \frac{1}{Q}, \quad \text{for } 1\le j\le k.$$

On the other hand, Ω_1'' can be estimated as

$$\begin{aligned}\Omega_1'' &\le \sum_{\substack{e_{11}\log p_1\le\log Q-\log r_1,\\ \vdots \\ e_{1k}\log p_k\le\log Q-\log r_1,}} \frac{r_1}{Q} \\ &\le \frac{r_1}{Q}\left(\frac{\log Q-\log r_1}{\log p_1}+1\right)\cdots\left(\frac{\log Q-\log r_1}{\log p_k}+1\right) \\ &\ll \frac{(\log Q)^k}{Q} \ll_\eta Q^{-1+\eta}.\end{aligned}$$

Combining all these estimates finally gives

$$\Omega_1 \ll_\eta Q^{-1+\eta}.$$

Similar estimates hold true for Ω_2, Ω_3 and Ω_4, and we have

$$A_{\delta,r_1,\dots,r_4}(Q) = A(\delta) + O_\eta\left(Q^{-1+\eta}\right).$$

Returning to $\sum$ from (5) and also to $S_{Q,\mathbf{I},H,G}$, we have in conclusion that

$$\begin{aligned} S_{Q,\mathbf{I},H,G} &= \frac{6cQ^4}{\pi^2}\left(\int_0^1 G(z)^2 \mathrm{d}z\right) \sum_{mr_1r_2r_3r_4\delta \le C_\wedge} \frac{\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)}{r_1r_2r_3r_4} \times \\ & \qquad c(\delta)A(\delta)\delta \iint_{[0,1]^4} H\left(\frac{9m\delta r_1r_2r_3r_4}{\pi^4 xyzw}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w + E, \end{aligned}$$

where, by (64)

$$E \ll_\eta Q^{4-\frac{1}{18}+\eta} + Q^{3+\eta} \ll_\eta Q^{4-\frac{1}{18}+\eta},$$

for any $0 < \eta < \frac{1}{18}$.

3.7. **Completion of the proof.** To complete the proof of Lemma 3 and to arrive at the simple formula promised for the pair correlation function in Theorem 1, we need to do further calculations. Recall that $C_\wedge = \frac{\pi^4 \wedge}{9}$ and write $S_{Q,\mathbf{I},H,G}$ as

$$S_{Q,\mathbf{I},H,G} = \frac{6cQ^4}{\pi^2}\left(\int_0^1 G(z)^2 \mathrm{d}z\right) S_\wedge + E, \tag{72}$$

where

$$S_\wedge = \sum_{1 \le k \le \frac{\pi^4 \wedge}{9}} \frac{H_k}{k} \sum_{mr_1r_2r_3r_4\delta = k} \frac{k\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)}{r_1r_2r_3r_4} c(\delta)A(\delta)\delta, \tag{73}$$

and

$$H_k = \iint_{[0,1]^4} H\left(\frac{9k}{\pi^4 xyzw}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w.$$

First, for fixed k with $1 \le k \le C_\wedge$, let

$$\Omega = [0,1]^4 \bigcap \left\{(x,y,z,w) \in \mathbb{R}^4 : xyzw > \frac{9k}{\pi^4 \wedge}\right\}.$$

Changing variables by $x' = \frac{9k}{\pi^4 xyzw}$, $y' = y$, $z' = z$, $w' = w$, Ω is mapped to the region

$$\Omega' = \left\{(x',y',z',w') \in \mathbb{R}^4 : 0 \le y', z', w' \le 1,\ \frac{9k}{\pi^4 y'z'w'} \le x' < \wedge\right\}.$$

Using the Jacobian of the transformation

$$\left|\frac{\partial(x,y,z,w)}{\partial(x',y',z',w')}\right| = \frac{9k}{\pi^4 x'^2 y'z'w'},$$

we have

$$H_k = \iint_\Omega H\left(\frac{9k}{\pi^4 xyzw}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w = \iint_{\Omega'} H(x') \frac{\frac{9k}{\pi^4}}{y'z'w'} \frac{1}{x'^2} \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,\mathrm{d}w'.$$

Changing the dummy variables x', y', z', w' back to x, y, z, w, this further gives

$$H_k = \frac{9k}{\pi^4} \int_{\frac{9k}{\pi^4}}^{\wedge} \frac{H(x)}{x^2} U\left(\frac{9k}{\pi^4 x}\right) \mathrm{d}x\,,$$

where the function $U : (0,1] \to [0,\infty)$ is given by

$$\begin{aligned} U(t) &= \int_t^1 \int_{\frac{t}{y}}^1 \int_{\frac{t}{yz}}^1 \frac{\mathrm{d}w}{w} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}y}{y} = \int_t^1 \int_{\frac{t}{y}}^1 \left(\log z + \log \frac{y}{t}\right) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}y}{y} \\ &= \int_t^1 \frac{1}{2} \log^2\left(\frac{y}{t}\right) \frac{\mathrm{d}y}{y} = \frac{1}{6} \log^3\left(\frac{1}{t}\right). \end{aligned}$$

Thus

$$H_k = \frac{3k}{2\pi^4} \int_{\frac{9k}{\pi^4}}^{\wedge} \frac{H(x)}{x^2} \log^3 \frac{\pi^4 x}{9k} \,\mathrm{d}x.$$

Next, we define a multiplicative function B by letting $B(n) = c(n)A(n)n^2$ for any $n \in \mathbb{N}$ and a multiplicative function ψ by the convolution

$$\psi = \mu * \mu * \mu * \mu * I_d * B,$$

where I_d is the identity function. It is easy to see that

$$\begin{aligned} &\sum_{m r_1 r_2 r_3 r_4 \delta = k} \frac{k\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)}{r_1 r_2 r_3 r_4} c(\delta)A(\delta)\delta \\ &= \sum_{m r_1 r_2 r_3 r_4 \delta = k} m\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)c(\delta)A(\delta)\delta^2 \\ &= \mu * \mu * \mu * \mu * I_d * B = \psi(k). \end{aligned}$$

Returning to $S_\wedge$ from (72), we get

$$\begin{aligned} S_\wedge &= \sum_{1 \le k \le \frac{\pi^4 \wedge}{9}} \frac{3\psi(k)}{2\pi^4} \left(\int_{\frac{9k}{\pi^4}}^{\wedge} \frac{H(x)}{x^2} \log^3 \frac{\pi^4 x}{9k} \,\mathrm{d}x \right) \\ &= \frac{3}{2\pi^4} \left(\int_0^{\wedge} \frac{H(x)}{x^2} \sum_{1 \le k \le \frac{\pi^4 \wedge}{9}} \psi(k) \max\left\{0, \log^3 \frac{\pi^4 x}{9k}\right\} \mathrm{d}x \right) \\ &= \frac{3}{2\pi^4} \left(\int_0^{\wedge} \frac{H(x)}{x^2} \sum_{1 \le k \le \frac{\pi^4 x}{9}} \psi(k) \log^3 \frac{\pi^4 x}{9k} \,\mathrm{d}x \right). \end{aligned}$$

From (72) we finally obtain

$$S_{Q,\mathbf{I},H,G} = \frac{9Q^4}{\pi^4}\left(\int_0^1 G(z)^2 \mathrm{d}z\right)\left(\int_0^{\wedge} \frac{H(x)}{x^2}\sum_{1\le k\le \frac{\pi^4 x}{9}} \frac{c\psi(k)}{\pi^2}\log^3\frac{\pi^4 x}{9k}\,\mathrm{d}x\right)+E,$$

where

$$E \ll_\eta Q^{4-\frac{1}{18}+\eta}.$$

Our last step is to compute the Zeta-function for ψ explicitly. For any $k, m \ge 1$, define a function H by

$$H(k,m) = \sum_{\substack{0\le n_1,n_2,n_3,n_4\le k\\ n_1+n_2+n_3+n_4=m}} 1.$$

By the definition of A from (71), we see that for any prime p and integer $m \ge 1$,

$$\begin{aligned}
A(p^m) &= \sum_{\substack{e_1,e_2,e_3,e_4\ge 0\\ e_1+e_2+e_3+e_4-\max\{e_1,e_2,e_3,e_4\}=m}} \frac{1}{p^{e_1+e_2+e_3+e_4}}\\
&= \sum_{k=1}^{\infty}\frac{1}{p^{k+m}}\sum_{\substack{e_1+e_2+e_3+e_4=k+m\\ \max\{e_1,e_2,e_3,e_4\}=k}} 1\\
&= \sum_{k=1}^{\infty}\frac{1}{p^{k+m}}\left(\sum_{\substack{0\le e_1,e_2,e_3,e_4\le k\\ e_1+e_2+e_3+e_4=k+m}} 1 - \sum_{\substack{0\le e_1,e_2,e_3,e_4\le k-1\\ e_1+e_2+e_3+e_4=k+m}} 1\right)\\
&= \sum_{k=1}^{\infty}\frac{H(k,k+m)-H(k-1,k+m)}{p^{k+m}}.
\end{aligned}$$

Then for $s\in\mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large and for any prime number p,

$$\begin{aligned}
H_p(s) &= 1+\sum_{m=1}^{\infty}\frac{B(p^m)}{p^{ms}} = 1+\sum_{m=1}^{\infty}\frac{p^{2m}\left(1-\frac{4}{p(p+3)}\right)A(p^m)}{p^{ms}}\\
&= 1+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{p^{2m}\left(1-\frac{4}{p(p+3)}\right)(H(k,k+m)-H(k-1,k+m))}{p^{ms}p^{k+m}}\\
&= 1+\left(1-\frac{4}{p(p+3)}\right)\sum_{k=1}^{\infty}p^{k(s-2)}\sum_{m=k+1}^{\infty}\left(\frac{H(k,m)}{p^{m(s-1)}}-\frac{H(k-1,m)}{p^{m(s-1)}}\right).
\end{aligned}$$

It is easy to see that

$$\sum_{m=0}^{\infty}H(k,m)q^m = \sum_{0\le n_1,n_2,n_3,n_4\le k} q^{n_1+n_2+n_3+n_4} = \left(\frac{1-q^{k+1}}{1-q}\right)^4,$$

for $0 \le m \le k$,

$$H(k,m) = \sum_{\substack{n_1,n_2,n_3,n_4 \ge 0 \\ n_1+n_2+n_3+n_4=m}} 1 = \binom{m+3}{3},$$

and for $k \ge 1$,

$$H(k,k) - H(k-1,k) = \sum_{\substack{0 \le n_1,n_2,n_3,n_4 \le k \\ n_1+n_2+n_3+n_4=k \\ \max\{n_1,n_2,n_3,n_4\}=k}} 1 = 4.$$

Therefore, denoting $q = p^{1-s}$, the inner sum of $H_p(s)$ on m is

$$\begin{aligned}
&\left(\sum_{m=0}^{\infty} \frac{H(k,m)}{p^{m(s-1)}} - \sum_{m=0}^{k} \frac{H(k,m)}{p^{m(s-1)}}\right) - \left(\sum_{m=0}^{\infty} \frac{H(k-1,m)}{p^{m(s-1)}} - \sum_{m=0}^{k} \frac{H(k-1,m)}{p^{m(s-1)}}\right) \\
= &\left(\left(\frac{1-q^{k+1}}{1-q}\right)^4 - \sum_{m=0}^{k} \binom{m+3}{3} q^m\right) - \\
&\left(\left(\frac{1-q^{k}}{1-q}\right)^4 - \sum_{m=0}^{k-1} \binom{m+3}{3} q^m - H(k-1,k)q^k\right) \\
= &\left(\frac{1-q^{k+1}}{1-q}\right)^4 - \left(\frac{1-q^{k}}{1-q}\right)^4 - 4q^k.
\end{aligned}$$

Replacing q by p^{1-s} we obtain that

$$\begin{aligned}
H_p(s) &= 1 + \left(1 - \frac{4}{p(p+3)}\right) \sum_{k=1}^{\infty} p^{k(s-2)} \times \\
&\quad \left(\left(\frac{1-p^{(1-s)(k+1)}}{1-p^{1-s}}\right)^4 - \left(\frac{1-p^{(1-s)k}}{1-p^{1-s}}\right)^4 - 4p^{(1-s)k}\right) \\
&= 1 + \frac{(p-1)(p+4)}{p(p+3)} \left\{ -\frac{4}{p-1} + \sum_{k=1}^{\infty} \frac{\left(1-p^{(1-s)(k+1)}\right)^4 - \left(1-p^{(1-s)k}\right)^4}{(1-p^{1-s})p^{k(2-s)}} \right\}.
\end{aligned}$$

Since

$$\psi = \mu * \mu * \mu * \mu * I_d * B,$$

it is clear that

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta^4(s)} \prod_{p \text{ prime}} H_p(s).$$

This completes the proof of Lemma 3, and therefore also the proof of Theorem 1.

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