ARITHMETIC MEAN OF DIFFERENCES OF DEDEKIND SUMS

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Abstract. Recently Girstmair and Schoissengeier studied the asymptotic behavior of the arithmetic mean of Dedekind sums

$$\frac{1}{\varphi(N)} \sum_{\substack{0 \le m < N \\ \gcd(m,N)=1}} |S(m,N)|,$$

as $N \to \infty$. In this paper we consider the arithmetic mean of weighted differences of Dedekind sums in the form

$$A_h(Q) = \frac{1}{\sum_{\frac{a}{q} \in \mathscr{F}_Q} h\left(\frac{a}{q}\right)} \times \sum_{\frac{a}{q} \in \mathscr{F}_Q} h\left(\frac{a}{q}\right) |s(a', q') - s(a, q)|,$$

where $h : [0,1] \to \mathbb{C}$ is a continuous function with $\int_0^1 h(t)\,\mathrm{d}t \neq 0$, $\frac{a}{q}$ runs over $\mathscr{F}_Q$, the set of Farey fractions of order Q in the unit interval $[0,1]$ and $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements of $\mathscr{F}_Q$. We show that the limit $\lim_{Q\to\infty} A_h(Q)$ exists and is independent of h.

1. Introduction

For any real number x, let $((x))$ be the sawtooth function defined as

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For positive integers h, k the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{s \pmod k} \left(\left(\frac{s}{k}\right)\right) \left(\left(\frac{hs}{k}\right)\right),$$

2000 *Mathematics Subject Classification.* 11F20.

Key words and phrases. Dedekind sums, Kloosterman sums, Farey fractions.

where the notation $s \pmod{k}$ means that s runs over a complete residue system modulo k. Since the sawtooth function has period one, $s(h,k)$ is a periodic function of h with period k.

The distribution of Dedekind sums, in particular the asymptotic behavior of even moments of such sums has been investigated by a number of authors. Recently Girstmair and Schoissengeier([5]) succeeded in establishing the right size of the more subtle first moment, which is the arithmetic mean

$$\frac{1}{\varphi(N)} \sum_{\substack{0 \le m < N \\ \gcd(m,N)=1}} |S(m,N)|,$$

as $N \to \infty$, where $S(m,N) = 12 \cdot s(m,N)$. In the process, they proved the asymptotic formula

$$\frac{1}{\varphi(N)} \sum_{\substack{m \in \mathcal{F} \\ \gcd(m,N)=1}} |S(m,N)| = \frac{3}{\pi^2} \log^2 N + O\left(\log^2 N / \log\log N\right),$$

as $N \to \infty$, where

$$\mathcal{F} = \bigcup_{1 \le d \le x} \bigcup_{\substack{0 \le c \le d \\ \gcd(c,d)=1}} I_{c/d} \subset [0,N),$$

$$I_{c/d} = [0,N] \bigcap \{z \in \mathbb{R} : |z - N \cdot c/d| \le x/d\},$$

$$x = \min\{\sqrt{N}/\log N, \sqrt{N}/\tau(N)\},$$

and $\tau(N)$ denotes the number of divisors of N. The sign changes and zones of large and small values for Dedekind sums which also sparked interest have been studied by Girstmair in ([3], [4]).

In this paper, we consider the arithmetic mean of weighted differences of Dedekind sums of the form $|s(a',q') - s(a,q)|$ with a weight function h, where $(\frac{a}{q}, \frac{a'}{q'})$ runs over the set of pairs of consecutive elements of the Farey sequence $\mathscr{F}_Q$ of order Q. The Farey sequence of order Q consists of all the fractions $\frac{a}{q} \in [0,1]$, in reduced form, with denominator bounded by Q, arranged in increasing order, i.e.,

$$\mathscr{F}_Q := \left\{ \frac{a}{q} \in [0,1] : a, q \in \mathbb{Z},\ \gcd(a,q) = 1,\ 1 \le q \le Q \right\}.$$

For basic properties of Farey sequences, the reader may consult Hardy and Wright [6]. We remark that in the limit as $Q \to \infty$, the above arithmetic mean turns out to be independent of the choices of weight h. More precisely one has the following result.

Theorem 1. *Let $h : [0,1] \to \mathbb{C}$ be a continuous function with $\int_0^1 h(t)\,\mathrm{d}t \neq 0$. Define*

$$A_h(Q) = \frac{1}{\sum_{\frac{a}{q} \in \mathscr{F}_Q} h\left(\frac{a}{q}\right)} \times \sum_{\frac{a}{q} \in \mathscr{F}_Q} h\left(\frac{a}{q}\right) |s(a', q') - s(a, q)|,$$

where $\frac{a}{q}$ runs over $\mathscr{F}_Q$, the set of Farey fractions of order Q in the unit interval $[0,1]$ and $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements of $\mathscr{F}_Q$. Then we have

$$\lim_{Q \to \infty} A_h(Q) = \frac{\sqrt{5} - 1}{12}.$$

We remark that the statement of the theorem holds more generally for piecewise continuous functions. In particular, by taking h to be the characteristic function of a subinterval $\mathbf{I}$ of $[0,1]$, we obtain the following result.

Corollary 1. *For any subinterval $\mathbf{I}$ of $[0,1]$, we have*

$$\lim_{Q \to \infty} \frac{1}{\#(\mathscr{F}_Q \bigcap \mathbf{I})} \sum_{\frac{a}{q} \in \mathscr{F}_Q \bigcap \mathbf{I}} |s(a', q') - s(a, q)| = \frac{\sqrt{5} - 1}{12}.$$

Acknowledgement. The authors are very grateful to the referee for many useful comments and suggestions.

2. Proof of theorem 1

Since any continuous function $h:[0,1]\to\mathbb{C}$ can be approximated uniformly by functions which are continuously differentiable, it is enough to prove the theorem in the case when h is continuously differentiable. In this case, we have the following stronger form of the result with a precise error term.

Theorem 1'. *Let $h:[0,1]\to\mathbb{C}$ be a continuously differentiable function with $\int_0^1 h(t)\,\mathrm{d}t\neq 0$. Define*

$$A_h(Q)=\frac{1}{\sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right)}\times\sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right)|s(a',q')-s(a,q)|,$$

where $\frac{a}{q}$ runs over $\mathscr{F}_Q$, the set of Farey fractions of order Q in the unit interval $[0,1]$ and $\frac{a}{q}<\frac{a'}{q'}$ are consecutive elements of $\mathscr{F}_Q$. Then for any fixed positive real number δ, we have

$$A_h(Q)=\frac{\sqrt{5}-1}{12}+O_{h,\delta}\left(Q^{-\frac{1}{16}+\delta}\right),$$

as $Q\to\infty$.

Proof of Theorem 1'. Our first objective is to obtain an asymptotic formula for $B_h(Q)$, where

$$B_h(Q)=\sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right)|s(a',q')-s(a,q)|.$$

Let $\frac{a}{q}<\frac{a'}{q'}$ be consecutive Farey fractions. We know that $a'q-aq'=1$. By using formula (38) on Page 29 of [8], which is a consequence of the reciprocity law of Dedekind sums, one has

$$s(a',q')-s(a,q)=s(q,q')+s(q',q)=-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right).$$

Therefore

$$B_h(Q) = \sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right)\left|-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\right|.$$

Define

$$B_{1,h}(Q) = \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q\\ q'\le q}} h\left(\frac{a}{q}\right)\left|-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\right|,$$

$$B_{2,h}(Q) = \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q\\ q'> q}} h\left(\frac{a}{q}\right)\left|-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\right|,$$

so that we have

$$B_h(Q) = B_{1,h}(Q) + B_{2,h}(Q).$$

By symmetry, it suffices to consider $B_{1,h}(Q)$ only. Clearly, the condition

$$-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\le 0,$$

is equivalent to $q^2+q'^2-3qq'+1\le 0$. Since q,q' are integers, this is equivalent to $q^2+q'^2-3qq'<0$, and we have $\frac{3-\sqrt{5}}{2}<\frac{q'}{q}<\frac{3+\sqrt{5}}{2}$. Therefore one has

$$-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right) \text{ is } \begin{cases} >0, & \frac{q'}{q}\le\frac{3-\sqrt{5}}{2} \text{ or } \frac{q'}{q}\ge\frac{3+\sqrt{5}}{2},\\ \le 0, & \frac{3-\sqrt{5}}{2}<\frac{q'}{q}<\frac{3+\sqrt{5}}{2}.\end{cases}$$

We can separate $B_{1,h}(Q)$ into two parts as $B_{1,h}(Q)=I+II$, where

$$I = \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q\\ \frac{3-\sqrt{5}}{2}<\frac{q'}{q}\le 1}} h\left(\frac{a}{q}\right)\left(\frac{1}{4}-\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\right),$$

$$II = \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q\\ \frac{q'}{q}\le\frac{3-\sqrt{5}}{2}}} h\left(\frac{a}{q}\right)\left(-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}+\frac{1}{qq'}\right)\right).$$

2.1. **Estimation for I.** It is known that for any two consecutive Farey fractions $\frac{a}{q} < \frac{a'}{q'}$ of order Q, one has $a'q - aq' = 1$ and $q + q' > Q$. Hence $aq' \equiv -1 \pmod{q}$ and $a \equiv -\bar{q'} \pmod{q}$, where $\bar{q'}$ is the multiplicative inverse of q' modulo q with $1 \leq \bar{q'} \leq q$ (here $\bar{q'}$ exists because $\gcd(q, q') = 1$). Since $1 \leq a < q$, one has $a = q - \bar{q'}$. Conversely, if q and q' are two coprime integers in $\{1, \ldots, Q\}$ with $q + q' > Q$, then there are unique $a \in \{1, \ldots, Q\}$ and $a' \in \{1, \ldots, Q\}$ for which $a'q - aq' = 1$, and $\frac{a}{q} < \frac{a'}{q'}$ are consecutive Farey fractions of order Q. We find that

$$I = \sum_{q \leq Q} \sum_{\substack{Q-q<q' \leq Q \\ \frac{3-\sqrt{5}}{2} q < q' \\ \gcd(q,q')=1}} h\left(1 - \frac{\bar{q'}}{q}\right)\left(\frac{1}{4} - \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right)\right).$$

The restrictions $Q - q < q' < q$ implies that $q > \frac{Q}{2}$ and when $Q - q = \frac{3-\sqrt{5}}{2} q$, we have $q = \left(\frac{5-\sqrt{5}}{2}\right)^{-1} \cdot Q = \frac{5+\sqrt{5}}{10} Q$, hence we can decompose I into two parts as

$$I = I_1 + I_2,$$

where

$$I_1 = \sum_{\frac{Q}{2} < q \leq \frac{5+\sqrt{5}}{10} Q} \sum_{\substack{Q-q<q' \leq q \\ \gcd(q,q')=1}} h\left(1 - \frac{\bar{q'}}{q}\right)\left(\frac{1}{4} - \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right)\right),$$

$$I_2 = \sum_{\frac{5+\sqrt{5}}{10} Q < q \leq Q} \sum_{\substack{\frac{3-\sqrt{5}}{2} Q < q' \leq q \\ \gcd(q,q')=1}} h\left(1 - \frac{\bar{q'}}{q}\right)\left(\frac{1}{4} - \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right)\right).$$

Sums of the above type can be estimated by the aid of the following two lemmas. The first lemma, whose proof depends on Weil-type bounds for Kloosterman sums, provides an asymptotic formula for certain sums over visible lattice points in planar domains satisfying congruence constraints.

Lemma 1. ([2], Lemma 2.2) *Assume that $q \geq 1$ is an integer, $\mathcal{I}_1$ and $\mathcal{I}_2$ are intervals with $|\mathcal{I}_1|, |\mathcal{I}_2| < q$, and $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is a C^1 function. Write $Dg = |\frac{\partial g}{\partial x}| + |\frac{\partial g}{\partial y}|$ and let $||\cdot||_\infty$ denote the L^∞ norm on $\mathcal{I}_1 \times \mathcal{I}_2$. Then for any integer $T > 1$, one has*

$$\sum_{\substack{a\in\mathcal{I}_1, b\in\mathcal{I}_2 \\ ab\equiv 1 \pmod q}} g(a,b) = \frac{\phi(q)}{q^2} \iint_{\mathcal{I}_1\times\mathcal{I}_2} g(x,y)\,\mathrm{d}x\mathrm{d}y + E_{q,\mathcal{I}_1,\mathcal{I}_2,g,T} \quad ,$$

where, for all $\delta > 0$,

$$E_{q,\mathcal{I}_1,\mathcal{I}_2,g,T} \ll_\delta T^2 q^{\frac{1}{2}+\delta} ||g||_\infty + T q^{\frac{3}{2}+\delta} ||Dg||_\infty + \frac{|\mathcal{I}_1||\mathcal{I}_2| \cdot ||Dg||_\infty}{T}.$$

In applying Lemma 1, we will also make use of the following result.

Lemma 2. ([1], Lemma 2.3) *Suppose that $0 < a < b$ are real numbers and that f is a C^1 function on $[a, b]$. Then*

$$\sum_{a<k\leq b} \frac{\phi(k)}{k} f(k) = \frac{6}{\pi^2} \int_a^b f(x)\,\mathrm{d}x + O\left(\log b \left(||f||_\infty + \int_a^b |f'(x)|\,\mathrm{d}x\right)\right).$$

For fixed q with $\frac{Q}{2} < q \leq \frac{5+\sqrt{5}}{10}Q$, we may apply Lemma 1 directly to I_1, with $\mathcal{I}_1 = [0, q], \mathcal{I}_2 = (Q-q, q], a = \bar{q'}, b = q'$ and

$$g(x,y) = h\left(1 - \frac{x}{q}\right)\left(\frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right)\right).$$

First of all, both $|\mathcal{I}_1|, |\mathcal{I}_2| \leq q \leq Q$. Moreover, under the restrictions $\frac{Q}{2} < q \leq \frac{5+\sqrt{5}}{2}Q$, $q \asymp Q$, and for $y \in \mathcal{I}_2$, we have

$$\frac{5-\sqrt{5}}{10}Q \leq Q - q < y \leq q \leq \frac{5+\sqrt{5}}{10}Q.$$

Hence $y \asymp Q$, and we have

$$\frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right) \leq \frac{1}{4},$$

$$\frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right) \geq -\frac{1}{12}\frac{1}{qy},$$

and

$$\frac{1}{qy} \asymp \frac{1}{Q^2}.$$

Therefore $\left|\frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right)\right| \ll 1$, so that we have $||g||_\infty \ll ||h||_\infty \ll_h 1$. Next, under the same restrictions one obtains

$$\left|\frac{\partial g}{\partial x}\right| \ll ||Dh||_\infty \frac{1}{q} \ll_h \frac{1}{Q},$$

$$\left|\frac{\partial g}{\partial y}\right| \le ||h||_\infty \frac{1}{12}\left(\frac{q}{y^2} + \frac{1}{q} + \frac{1}{qy^2}\right) \ll_h \frac{1}{Q}.$$

Lemma 1 gives us that for any integer $T > 1$,

$$I_1 = \sum_{\frac{Q}{2} < q \le \frac{5+\sqrt{5}}{10} Q} \left\{ \frac{\phi(q)}{q^2} \iint_{\mathcal{I}_1 \times \mathcal{I}_2} h\left(1 - \frac{x}{q}\right)\left(\frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right)\right) \mathrm{d}x\mathrm{d}y + E_{q,h,T} \right\},$$

where for all $\delta > 0$,

$$E_{q,h,T} \ll_{\delta,h} T^2 q^{\frac{1}{2}+\delta} + T q^{\frac{3}{2}+\delta} \frac{1}{Q} + \frac{Q^2 \cdot \frac{1}{Q}}{T}.$$

The double integral over $\mathcal{I}_1 \times \mathcal{I}_2$ is

$$\begin{aligned} &\int_0^q h\left(1 - \frac{x}{q}\right) \mathrm{d}x \int_{Q-q}^q \frac{1}{4} - \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q} + \frac{1}{qy}\right) \mathrm{d}y \\ &= q^2 \int_0^1 h(t)\,\mathrm{d}t \int_{\frac{Q}{q}-1}^1 \frac{1}{4} - \frac{1}{12}\left(y + \frac{1}{y} + \frac{1}{q^2 y}\right) \mathrm{d}y. \end{aligned}$$

Therefore

$$I_1 = \left(\int_0^1 h(t)\,\mathrm{d}t\right) \sum_{\frac{Q}{2} < q \le \frac{5+\sqrt{5}}{10} Q} \frac{\phi(q)}{q} q \int_{\frac{Q}{q}-1}^1 \frac{1}{4} - \frac{1}{12}\left(y + \frac{1}{y} + \frac{1}{q^2 y}\right) \mathrm{d}y + E'_{h,T},$$

where

$$E'_{h,T} \ll Q \cdot E_{q,h,T} \ll_{\delta,h} T^2 Q^{\frac{3}{2}+\delta} + T Q^{\frac{3}{2}+\delta} + \frac{Q^2}{T}.$$

Let $T^2 Q^{\frac{3}{2}} \approx \frac{Q^2}{T}$, we may choose $T \approx Q^{\frac{1}{6}}$ to obtain

$$E'_{h,T} \ll_{\delta,h} Q^{2-\frac{1}{6}+\delta}. \tag{1}$$

Since

$$\sum_{\frac{Q}{2}<q\leq\frac{5+\sqrt{5}}{10}Q}\frac{\phi(q)}{q}q\int_{\frac{Q}{q}-1}^{1}\frac{1}{q^2y}\,\mathrm{d}y \ll \sum_{\frac{Q}{2}<q\leq\frac{5+\sqrt{5}}{10}Q}\frac{\phi(q)}{q^2} \ll \sum_{\frac{Q}{2}<q\leq\frac{5+\sqrt{5}}{10}Q}\frac{1}{Q} \ll 1,$$

we still have

$$I_1 \;=\; \left(\int_0^1 h(t)\,\mathrm{d}t\right)\sum_{\frac{Q}{2}<q\leq\frac{5+\sqrt{5}}{10}Q}\frac{\phi(q)}{q}q\int_{\frac{Q}{q}-1}^{1}\frac{1}{4}-\frac{1}{12}\left(y+\frac{1}{y}\right)\mathrm{d}y + E'_{h,T}.$$

Now applying Lemma 2 to I_1, with the function

$$(2)\qquad f(x) = x\int_{\frac{Q}{x}-1}^{1}\frac{1}{4}-\frac{1}{12}\left(y+\frac{1}{y}\right)\mathrm{d}y,\quad \frac{Q}{2}<x\leq\frac{5+\sqrt{5}}{10}Q,$$

and $a=\frac{Q}{2}, b=\frac{5+\sqrt{5}}{10}Q$, we have

$$(3)\qquad \sum_{\frac{Q}{2}<q\leq\frac{5+\sqrt{5}}{10}Q}\frac{\phi(q)}{q}f(q) = \frac{6}{\pi^2}\int_{\frac{Q}{2}}^{\frac{5+\sqrt{5}}{2}Q} f(x)\,\mathrm{d}x + E'.$$

Here for the error term E', notice that for $\frac{Q}{2} < x \leq \frac{5+\sqrt{5}}{10}Q$, we have $||f||_\infty \ll Q$ and by the chain rule it is easy to see that

$$|f'(x)| \ll 1 + x\frac{Q}{x^2} \ll 1.$$

Hence $\int_{\frac{Q}{2}}^{\frac{5+\sqrt{5}}{10}} |f'(x)|\,\mathrm{d}x \ll Q$, and Lemma 2 yields

$$(4)\qquad E' \ll Q\log Q \ll_\delta Q^{1+\delta},$$

for any $\delta > 0$. Putting (4),(3) and (1) together and returning to I_1 we get

$$I_1 \;=\; \frac{6}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\int_{\frac{Q}{2}}^{\frac{5+\sqrt{5}}{10}Q} f(x)\,\mathrm{d}x + O_{\delta,h}\left(Q^{2-\frac{1}{6}+\delta}\right).$$

Finally, writing $f(x)$ explicitly in (2) and using the change of variable $\frac{x}{Q} = x'$, we obtain as $Q\to\infty$ the asymptotic formula

$$I_1 \;=\; \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot C_1 + O_{\delta,h}\left(Q^{2-\frac{1}{6}+\delta}\right)$$

for any fixed positive real number δ, where the constant C_1 is given by

$$C_1 = \int_{\frac{1}{2}}^{\frac{5+\sqrt{5}}{10}} x \int_{\frac{1}{x}-1}^{1} \frac{1}{4} - \frac{1}{12}\left(y + \frac{1}{y}\right) \mathrm{d}y \, \mathrm{d}x. \tag{5}$$

Following exactly the same procedure we can get a similar asymptotic formula for I_2 as

$$I_2 = \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right) \cdot C_2 + O_{\delta,h}\left(Q^{2-\frac{1}{6}+\delta}\right),$$

where the constant C_2 is given by

$$C_2 = \int_{\frac{5+\sqrt{5}}{10}}^{1} x \int_{\frac{3-\sqrt{5}}{2}}^{1} \frac{1}{4} - \frac{1}{12}\left(y + \frac{1}{y}\right) \mathrm{d}y \, \mathrm{d}x. \tag{6}$$

Therefore for any fixed positive real number δ, as $Q \to \infty$, we have

$$I = I_1 + I_2 = \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right) \cdot (C_1 + C_2) + O_{\delta,h}\left(Q^{2-\frac{1}{6}+\delta}\right). \tag{7}$$

The constants C_1, C_2 can be computed separately but the expressions are complicated. Nevertheless it turns out that $C_1 + C_2$ has a simple form as $C_1 + C_2 = \frac{\sqrt{5}-1}{96}$.

2.2. **Estimation for *II*.** We treat II similarly, but with a slight difference. Take a number K between 0 and Q which will be chosen later and denote

$$II' = \sum_{\substack{\frac{a}{q} \in \mathscr{F}_Q \\ \frac{q'}{q} \le \frac{3-\sqrt{5}}{2} \\ q' \ge K}} h\left(\frac{a}{q}\right)\left(-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right)\right),$$

$$II'' = \sum_{\substack{\frac{a}{q} \in \mathscr{F}_Q \\ \frac{q'}{q} \le \frac{3-\sqrt{5}}{2} \\ q' < K}} h\left(\frac{a}{q}\right)\left(-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right)\right),$$

so that

$$II = II' + II''.$$

For II'', as we know,

$$|II''| \ll_h \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ \frac{q'}{q}\le\frac{3-\sqrt{5}}{2} \\ q'<K}} \left(1+\frac{q'}{q}+\frac{1}{qq'}\right) + \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ q'<K}} \frac{q}{q'},$$

where the first term is

$$\ll_h \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ q'<K}} 1 = \sum_{1\le q'<K} \phi(q') \ll K^2,$$

and the second term is

$$\ll_h Q \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ q'<K}} \frac{1}{q'} = Q \sum_{1\le q'<K} \frac{\phi(q')}{q'} \le Q \sum_{1\le q'<K} 1 \ll QK.$$

Since $K<Q$, we obtain that

$$II'' \ll_h QK.$$

For II', first notice that

$$\sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ \frac{q'}{q}\le\frac{3-\sqrt{5}}{2} \\ q'\ge K}} \left|h\left(\frac{a}{q}\right)\right| \frac{1}{qq'} \ll_h \frac{1}{K} \sum_{\frac{a}{q}\in\mathscr{F}_Q} \frac{1}{q} = \frac{1}{K}\sum_{q\le Q}\frac{\phi(q)}{q} \ll \frac{Q}{K},$$

hence

$$II' = \sum_{\substack{\frac{a}{q}\in\mathscr{F}_Q \\ \frac{q'}{q}\le\frac{3-\sqrt{5}}{2} \\ q'\ge K}} h\left(\frac{a}{q}\right)\left(-\frac{1}{4}+\frac{1}{12}\left(\frac{q}{q'}+\frac{q'}{q}\right)\right) + O_h\left(\frac{Q}{K}\right).$$

For simplicity, we still denote the main term by II'. Now we follow the same procedure as for I_1 and I_2. Since $Q-q<q'\le\frac{3-\sqrt{5}}{2}q$, we have

$$q > \left(\frac{5-\sqrt{5}}{2}\right)^{-1}\cdot Q = \frac{5+\sqrt{5}}{10}Q,$$

and by the one-to-one correspondence between pairs of consecutive Farey fractions of order Q and coprime integers (q, q') with $1 \le q, q' \le Q, q + q' > Q$, we can write II' as

$$II' = \sum_{\frac{5+\sqrt{5}}{10}Q < q \le Q} \sum_{\substack{K \le q' \le \frac{3-\sqrt{5}}{2}q \\ Q-q<q' \\ 0 \le \bar{q'} \le q \\ q'\bar{q'} \equiv 1 \pmod q}} h\left(1 - \frac{\bar{q'}}{q}\right)\left(-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q}\right)\right).$$

We may apply Lemma 1 to II', with $\mathcal{I}_1 = [0, q]$, $\mathcal{I}_2 = (\max\{Q-q, K\}, \frac{3-\sqrt{5}}{2}q]$, $a = \bar{q'}, b = q'$ and

$$g(x, y) = h\left(1 - \frac{x}{q}\right)\left(-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q}\right)\right).$$

First notice that both $|\mathcal{I}_1|, |\mathcal{I}_2| \le q \le Q$, and since $q \asymp Q$, $y \in \mathcal{I}_2$ and $y \ge K$, we have

$$||g||_\infty \ll ||h||_\infty \frac{Q}{K} \ll_h \frac{Q}{K}.$$

Moreover as $q \asymp Q$, $y \in \mathcal{I}_2$ and $K < Q$, it follows that

$$\left|\frac{\partial g}{\partial x}\right| \le ||Dh||_\infty \frac{1}{q}\left(\frac{q}{K} + 1\right) \ll_h \frac{1}{K},$$

$$\left|\frac{\partial g}{\partial y}\right| \le ||h||_\infty \frac{1}{12}\left(\frac{q}{y^2} + \frac{1}{q}\right) \ll_h \frac{Q}{K^2} + \frac{1}{Q} \ll \frac{Q}{K^2}.$$

Applying Lemma 1, for any integer $T > 1$ we have

$$II' = \sum_{\frac{5+\sqrt{5}}{10}Q < q \le Q} \left\{\frac{\phi(q)}{q^2} \iint_{\mathcal{I}_1 \times \mathcal{I}_2} h\left(1 - \frac{x}{q}\right)\left(-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{y} + \frac{y}{q}\right)\right) \mathrm{d}x\mathrm{d}y + E_{q,h,T}\right\},$$

where for all $\delta > 0$,

$$E_{q,h,T} \ll_{\delta,h} T^2 q^{\frac{1}{2}+\delta}\frac{Q}{K} + Tq^{\frac{3}{2}+\delta}\frac{Q}{K^2} + \frac{Q^3}{TK^2}.$$

The integral inside is

$$\int_0^q h\left(1-\frac{x}{q}\right)\,\mathrm{d}x\int_{\max\{Q-q,K\}}^{\frac{3-\sqrt{5}}{2}q} -\frac{1}{4}+\frac{1}{12}\left(\frac{q}{y}+\frac{y}{q}\right)\,\mathrm{d}y$$
$$= q^2\int_0^1 h(t)\,\mathrm{d}t\int_{\max\{\frac{Q}{q}-1,\frac{K}{q}\}}^{\frac{3-\sqrt{5}}{2}} -\frac{1}{4}+\frac{1}{12}\left(y+\frac{1}{y}\right)\,\mathrm{d}y.$$

Therefore

$$II' = \left(\int_0^1 h(t)\,\mathrm{d}t\right)\sum_{\frac{5+\sqrt{5}}{10}Q<q\le Q}\frac{\phi(q)}{q}\times$$
$$q\int_{\max\{\frac{Q}{q}-1,\frac{K}{q}\}}^{\frac{3-\sqrt{5}}{2}} -\frac{1}{4}+\frac{1}{12}\left(y+\frac{1}{y}\right)\,\mathrm{d}y + E'_{h,T},$$

where

$$E'_{h,T}\ll Q\cdot E_{q,h,T}\ll_{\delta,h}\frac{T^2Q^{\frac{5}{2}+\delta}}{K}+\frac{TQ^{\frac{7}{2}+\delta}}{K^2}+\frac{Q^4}{TK^2}.$$

To minimize the error terms QK, $\frac{T^2Q^{\frac{5}{2}}}{K}$, $\frac{TQ^{\frac{7}{2}}}{K^2}$ and $\frac{Q^4}{TK^2}$, assume that

$$QK\approx\frac{T^2Q^{\frac{5}{2}}}{K}\approx\frac{Q^4}{TK^2},$$

and we may choose $K=Q^{1-\frac{1}{16}}, T\approx Q^{\frac{3}{16}}$. Consequently

$$E'_{h,T}\ll_{\delta,h} Q^{2-\frac{1}{16}+\delta}. \tag{8}$$

Next, we applying Lemma 2 to II' with function

$$f(x)=x\int_{\max\{\frac{Q}{x}-1,\frac{K}{x}\}}^{\frac{3-\sqrt{5}}{2}} -\frac{1}{4}+\frac{1}{12}\left(y+\frac{1}{y}\right)\,\mathrm{d}y, \tag{9}$$

and $a=\frac{5+\sqrt{5}}{10}Q, b=Q$. Notice that since $\frac{5+\sqrt{5}}{10}Q<x\le Q$, we have

$$||f||_\infty\ll Q\frac{Q}{K}\ll Q^{1+\frac{1}{16}}.$$

It is easy to see that

$$|f'(x)|\ll\frac{Q}{K}+Q\frac{Q}{K^2}\ll Q^{\frac{1}{8}}.$$

Therefore $\int_{\frac{5+\sqrt{5}}{10}Q}^{Q} |f'(x)|\,\mathrm{d}x \ll Q^{1+\frac{1}{8}}$. Then Lemma 2 gives

$$\sum_{\frac{5+\sqrt{5}}{10}Q<q\le Q} \frac{\phi(q)}{q} f(q) = \frac{6}{\pi^2}\int_{\frac{5+\sqrt{5}}{10}Q}^{Q} f(x)\,\mathrm{d}x + E', \tag{10}$$

where and

$$E' \ll \log Q\left(Q^{1+\frac{1}{16}} + Q^{1+\frac{1}{8}}\right) \ll_\delta Q^{1+\frac{1}{8}+\delta}, \tag{11}$$

for any $\delta > 0$. Putting (10),(11) and (8) together and returning to II' we obtain

$$II' \;=\; \frac{6}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\int_{\frac{5+\sqrt{5}}{10}Q}^{Q} f(x)\,\mathrm{d}x + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right).$$

Finally writing $f(x)$ explicitly in (9) and making the change of variable $\frac{x}{Q} = x'$, we get

$$II' \;=\; \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot C_{\frac{K}{Q}} + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right),$$

for any fixed positive real number δ, where the number $C_{\frac{K}{Q}}$ is given as

$$C_{\frac{K}{Q}} = \int_{\frac{5+\sqrt{5}}{10}}^{1} x\int_{\max\{\frac{1}{x}-1,\frac{K}{Qx}\}}^{\frac{3-\sqrt{5}}{2}} -\frac{1}{4}+\frac{1}{12}\left(y+\frac{1}{y}\right)\mathrm{d}y\,\mathrm{d}x.$$

Denote the constant C_3 by the integral

$$\begin{aligned} C_3 \;&=\; \int_{\frac{5+\sqrt{5}}{10}}^{1} x\int_{\frac{1}{x}-1}^{\frac{3-\sqrt{5}}{2}} -\frac{1}{4}+\frac{1}{12}\left(y+\frac{1}{y}\right)\mathrm{d}y\,\mathrm{d}x \\ &=\; \frac{\sqrt{5}-1}{96}. \end{aligned} \tag{12}$$

Since $K = Q^{1-\frac{1}{16}}$, for any positive real number δ we have

$$\begin{aligned}
|C_3 - C_{\frac{K}{Q}}| &= \int_{1-\frac{K}{Q}}^{1} x \int_{\frac{1}{x}-1}^{\frac{K}{Qx}} -\frac{1}{4} + \frac{1}{12}\left(y + \frac{1}{y}\right) \mathrm{d}y\,\mathrm{d}x \\
&= \int_{1-\frac{K}{Q}}^{1} x \int_{\frac{1}{x}-1}^{\frac{K}{Qx}} -\frac{1}{4} + \frac{y}{12} \mathrm{d}y\,\mathrm{d}x + \int_{1-\frac{K}{Q}}^{1} x \int_{\frac{1}{x}-1}^{\frac{K}{Qx}} \frac{1}{12y} \mathrm{d}y\,\mathrm{d}x \\
&\ll \frac{K}{Q} + \int_{1-\frac{K}{Q}}^{1} x\left(\log\frac{K}{Q} - \log(1-x)\right) \mathrm{d}x \\
&\ll \log\left(\frac{Q}{K}\right)\frac{K}{Q} \ll \frac{\log Q}{Q^{\frac{1}{16}}} \ll_{\delta} Q^{-\frac{1}{16}+\delta},
\end{aligned}$$

therefore

$$II' = \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot C_3 + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right),$$

and

$$II = II' + II'' = \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot C_3 + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right). \tag{13}$$

2.3. **Proof of Theorem 1'.** Putting the estimate (7) and (13) together, we have the asymptotic formula

$$\begin{aligned}
B_{1,h}(Q) &= I + II \\
&= \frac{6Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot(C_1 + C_2 + C_3) + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right),
\end{aligned}$$

for any positive real number δ, where the constants C_1, C_2, C_3 are defined in (5),(6) and (12) respectively. By symmetry we have exactly the same asymptotic formula for $B_{2,h}(Q)$. Therefore, we may denote the constant C by

$$C = 4(C_1 + C_2 + C_3) = \frac{\sqrt{5}-1}{12} \approx 0.103006. \tag{14}$$

It follows that

$$\begin{aligned} B_h(Q) &= B_{1,h}(Q) + B_{2,h}(Q) \\ &= \frac{3Q^2}{\pi^2}\left(\int_0^1 h(t)\,\mathrm{d}t\right)\cdot\frac{\sqrt{5}-1}{12} + O_{\delta,h}\left(Q^{2-\frac{1}{16}+\delta}\right). \end{aligned}$$

By using Koksma's Inequality([7]), one has

$$\frac{1}{\#(\mathscr{F}_Q)}\sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right) = \int_0^1 h(t)\,\mathrm{d}t + O_h\left(\frac{1}{Q}\right).$$

We also know that

$$\#(\mathscr{F}_Q) = \frac{3Q^2}{\pi^2} + O(Q\log Q).$$

Counting all these facts together we obtain as $Q\to\infty$,

$$A_h(Q) = \frac{B_h(Q)}{\sum_{\frac{a}{q}\in\mathscr{F}_Q} h\left(\frac{a}{q}\right)} = \frac{\sqrt{5}-1}{12} + O_{\delta,h}\left(Q^{-\frac{1}{16}+\delta}\right),$$

for any fixed positive real number δ. This completes the proof of Theorem $1'$.

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