Correlation of fractions with divisibility constraints

Maosheng Xiong[*1] and **Alexandru Zaharescu**[**2]

[1] 210 McAllister Building, Department of Mathematics, Eberly College of Science, Pennsylvania State University, University Park, PA 16802 USA

[2] Institute of Mathematics of the Romanian Academy, P.O. Box 1–764, 70700 Bucharest, Romania, and Department of Mathematics, University of Illinois at Urbana-Champaign, 273 Altgeld Hall, MC-382, 1409 W. Green Street, Urbana, IL 61801 USA

Key words Pair correlation, Farey fractions, Poisson distribution.
MSC (2000) 11J71, 11B57; 11K36

Let $B = (B_Q)_{Q\in\mathbb{N}}$ be an increasing sequence of positive square free integers satisfying the condition that $B_{Q_1}|B_{Q_2}$ whenever $Q_1 < Q_2$. For any subinterval $\mathbf{I} \subset [0,1]$, let

$$\mathscr{F}_{B,Q}(\mathbf{I}) = \left\{a/q \in \mathbf{I} : 1 \le a \le q \le Q, \gcd(a,q) = \gcd(q, B_Q) = 1\right\} .$$

It is shown that if $B_Q \ll Q^{\log\log Q/4}$, then the limiting pair correlation function of the sequence $\left(\mathscr{F}_{B,Q}(\mathbf{I})\right)_{Q\in\mathbb{N}}$ exists and is independent of the subinterval $\mathbf{I}$. Moreover, the sequence is Poissonian if $\lim_{Q\to\infty} \frac{\varphi(B_Q)}{B_Q} = 0$, and exhibits a very strong repulsion if $\lim_{Q\to\infty} \frac{\varphi(B_Q)}{B_Q} \neq 0$, where φ is Euler's totient function.

1 Introduction and statement of results

The statistics of spacings measure the fine structure of the distribution of a sequence in a more subtle way than the classical Weyl uniform distribution ([20]). Their study was initiated by physicists (see for example Wigner [21] and Dyson [5, 6, 7]) in order to understand the spectra of high energies. These notions have received a great deal of attention in many areas of mathematical physics, analysis, probability theory and number theory.

In the study of the spacing statistics of a sequence, correlations play an important role. There are few sequences of interest for which one could establish the existence of correlation measures, and many of them are conditional, as in the important case of the zeros of the Riemann zeta function, or more general L-functions ([11, 13, 17, 18]). In [3], Boca and one of the authors have established explicitly the pair correlation function of Farey fractions in the unit interval $[0,1]$ and their result shows a very strong repulsion between the elements of the sequence. In this paper, we consider more generally subsets of fractions with denominators satisfying certain divisibility constraints, and investigate the subtle effect these arithmetic constraints have on the pair correlation of the sequence.

The study of the distribution of Farey fractions is of independent interest. By the classical contributions of Franel and Landau ([8, 10], see also [15]), the existence of zero-free regions $1 - \delta_0 < \operatorname{Re} s < 1$ for the Riemann zeta function, and in particular the Riemann hypothesis, are equivalent to quantitative statements about the uniform distribution of Farey fractions. Ideas and techniques from this area, especially those which make use of Weil-Salié type estimates for Kloosterman sums, turned out to be useful in proving sharp asymptotic formulas in problems from various areas of mathematics ([1, 2, 4, 12]).

* Corresponding author: Maosheng Xiong, e-mail: xiong@math.psu.edu
** e-mail: zaharesc@math.uiuc.edu. The second author was supported by NSF grant number DMS-0456615, and by CNCSIS grant GR106/2007, code 1116, of the Romanian Ministry of Education and Research.

Let $\mathscr{F}$ be a finite set of cardinality N in $[0,1]$. The pair correlation measure $\mathcal{R}_{\mathscr{F}}(C)$ of a finite interval $C \subset \mathbb{R}$ is defined as

$$\frac{1}{N}\#\{(x,y) \in \mathscr{F}^2 : x \neq y, x-y \in \frac{1}{N}C + \mathbb{Z}\}.$$

Suppose that $(\mathscr{F}_n)_n$ is an increasing sequence of finite subsets of $[0,1]$ and that

$$\mathcal{R}(C) = \lim_{n\to\infty} \mathcal{R}_{\mathscr{F}_n}(C)$$

exists for every finite interval $C \subset \mathbb{R}$. Then $\mathcal{R}(C)$ is called the limiting pair correlation measure of $(\mathscr{F}_n)_n$. If

$$\mathcal{R}(C) = \int_C g(x)\,\mathrm{d}x,$$

then g is called the limiting pair correlation function of $(\mathscr{F}_n)_n$.

In this paper, we assume that $B = (B_Q)_{Q\in\mathbb{N}}$ is an increasing sequence of positive square-free integers satisfying $B_{Q_1}|B_{Q_2}$ whenever $Q_1 < Q_2$, and for any subinterval $\mathbf{I} \subset [0,1]$, we consider the pair correlation of the sequence $(\mathscr{F}_{B,Q}(\mathbf{I}))_{Q\in\mathbb{N}}$, where

$$\mathscr{F}_{B,Q}(\mathbf{I}) = \left\{a/q \in \mathbf{I} : 1 \leq a \leq q \leq Q, \gcd(a,q) = \gcd(q,B_Q) = 1\right\} .$$

While our result shows that the limiting pair correlation of the sequence $(\mathscr{F}_{B,Q}(\mathbf{I}))_Q$ exists and is independent of the subinterval $\mathbf{I}$ if the number of distinct prime divisors of B_Q is fairly small compared with the number of all primes less than Q, a surprising phenomenon appears, producing two different types of behavior depending on the limit of $\frac{\varphi(B_Q)}{B_Q}$ as Q goes to infinity. To state the main result precisely, we use the following notation. Define two functions δ, δ' by

$$\delta(n) = \prod_{p|n}(1+p^{-1}), \quad \delta'(n) = \prod_{p|n}(1-p^{-2}), \quad n \in \mathbb{N}. \tag{1.1}$$

Since $B_{Q_1}|B_{Q_2}$ whenever $Q_1 < Q_2$, both $\delta(B_Q)$ and $\delta'(B_Q)$ are non-negative monotonic functions as Q increases. The limit $\lim_{Q\to\infty}\delta'(B_Q)$ always exists, and we denote it by $\delta'(B)$. Let

$$\delta(B) = \lim_{Q\to\infty}\delta(B_Q),$$

if the limit exists, or let $\delta(B) = \infty$ otherwise. Denote by ω the function counting the number of distinct prime divisors. We prove the following result.

Theorem 1.1 *Assume that*

$$\omega(B_Q) \leq \frac{\log Q}{5\log 2} + O(1)\,. \tag{1.2}$$

Then for any subinterval $\mathbf{I}$ *of* $[0,1]$*, the limiting pair correlation function of the sequence* $(\mathscr{F}_{B,Q}(\mathbf{I}))_Q$ *exists and is independent of* $\mathbf{I}$. *Moreover, if* $\delta(B) = \infty$*, then the limiting pair correlation function is constant equal to* 1*. If* $\delta(B) < \infty$*, then the limiting pair correlation function is given by*

$$g_B(\lambda) = \frac{6\delta'(B)}{\pi^2\lambda^2\delta(B)^2}\sum_{1\leq k\leq\frac{\pi^2\lambda\delta(B)}{3}}\gcd(k,B_Q)\varphi\left(\frac{k}{\gcd(k,B_Q)}\right)\log\frac{\pi^2\lambda\delta(B)}{3k}, \tag{1.3}$$

for any $\lambda \geq 0$*. Here* Q *is chosen large enough in terms of* λ *and the sequence* B *and* φ *is Euler's totient function.*

We remark that, firstly, if $\delta(B) < \infty$ and $\lambda > 0$ is fixed, since there are only finitely many k with $1 \leq k \leq \frac{\pi^2\lambda\delta(B)}{3}$, and the values $\gcd(k,B_Q)$ become constant for Q sufficiently large, the function $g_B(\lambda)$ in (1.3) is well defined. Secondly, since $\omega(B_Q) \leq (1+o(1))\frac{\log B_Q}{\log\log B_Q}$ ([16]), if $B_Q \ll Q^{\log\log Q/4}$, the condition (1.2) will be

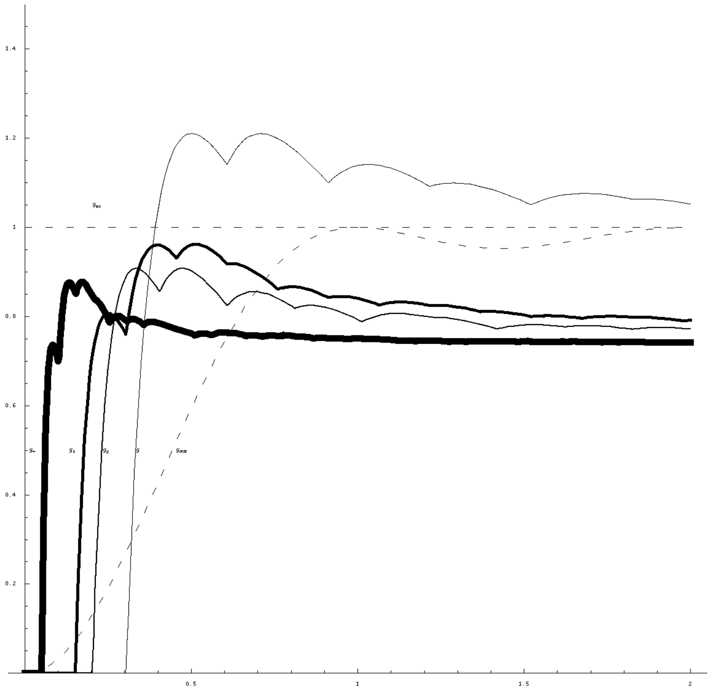

Fig. 1 Graphs of $g_1(\lambda)$, $g_2(\lambda)$, $g_6(\lambda)$, $g_\infty(\lambda)$, $g_{GUE}(\lambda)$ and g_{PO}

satisfied. Finally, if $\mathbf{I} = [0,1]$ and $B_Q = 1$ for any $Q \in \mathbb{N}$, then (1.3) reduces to the limiting pair correlation function provided in [3].

The graphs of g_1,g_2, g_6 with $B_Q = 1, 2, 6$ respectively and g_∞ with B_Q the product of the first 100 prime numbers are shown in Figure 1. For comparison, we also show the graphs of the pair correlation functions in the GUE and Poisson models, which are $g_{GUE}(\lambda) = 1 - \left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2$ and respectively $g_{PO} = 1$. All the functions g_1, g_2, g_6 and g_∞ vanish at zero, presenting a strong repulsion between the elements of the sequence. It is also clear from the graph that as B_Q contains more and more prime factors, the repulsion decreases and the distribution tends to become Poissonian.

In the paper, we will make use of the identity

$$\sum_{\gamma \in \mathscr{F}_{B,Q}([0,1])} e(r\gamma) = \sum_{\substack{d \leq Q \\ \gcd(d, B_Q) = 1}} \mu(d) \sum_{\substack{q \leq Q/d \\ \gcd(q, B_Q) = 1 \\ q \mid r}} q, \quad r \in \mathbb{Z}, \tag{1.4}$$

which can be derived from the Möbius inversion formula. Here the function $e(y) = \exp(2\pi i y)$ for any $y \in \mathbb{R}$. We first estimate the quantity $\#\mathscr{F}_{B,Q}(\mathbf{I})$ in Section 2, then we start the process of proving Theorem 1.1 in Section 3. In Section 4, to complete the proof, the two cases $\delta(B) < \infty$ and $\delta(B) = \infty$ are then treated separately.

2 Farey fractions in a short interval

For any subinterval $\mathbf{I} \subset [0,1]$, we need an asymptotic formula for the quantity $\#\mathscr{F}_{B,Q}(\mathbf{I})$. We first obtain the result for $\mathbf{I} = [0,1]$.

Lemma 2.1 *We have*

$$\#\mathscr{F}_{B,Q}([0,1]) = \frac{3Q^2}{\pi^2 \delta(B_Q)} + O\left(d(B_Q) Q \log Q\right),$$

as $Q \to \infty$, *where* d *is the divisor function, that is,* $d(n) = \sum_{r \mid n} 1$ *for any positive integer* n.

Proof. Using the Möbius inversion formula we have

$$\sum_{\substack{q\le X\\ \gcd(q,B_Q)=1}} q = \sum_{\substack{D|B_Q\\ D\le X}} \mu(D) \sum_{\substack{q\le X\\ D|q}} q = \sum_{\substack{D|B_Q\\ D\le X}} \mu(D) \sum_{q\le X/D} qD\,.$$

This becomes

$$\sum_{\substack{D|B_Q\\ D\le X}} \mu(D)D\frac{\left(\frac{X}{D}\right)^2 + O\left(\frac{X}{D}\right)}{2} = \frac{X^2}{2}\sum_{\substack{D|B_Q\\ D\le X}}\frac{\mu(D)}{D} + O(d(B_Q)X).$$

Then

$$\#\mathscr{F}_{B,Q}([0,1]) = \sum_{\substack{q\le Q\\ \gcd(q,B_Q)=1}} \varphi(q) = \sum_{\substack{q\le Q\\ \gcd(q,B_Q)=1}} q\sum_{d|q}\frac{\mu(d)}{d} = \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1}} q\,.$$

By using the above estimate, this can be simplified as

$$\frac{Q^2}{2}\sum_{\substack{D|B_Q\\ D\le Q}}\frac{\mu(D)}{D}\sum_{\substack{d\le Q/D\\ \gcd(d,B_Q)=1}}\frac{\mu(d)}{d^2} + O(d(B_Q)Q\log Q).$$

Since

$$\sum_{\substack{d\le Q/D\\ \gcd(d,B_Q)=1}}\frac{\mu(d)}{d^2} = \prod_{p\nmid B_Q}(1-p^{-2}) + O(D/Q),$$

$$\sum_{\substack{D|B_Q\\ D>Q}}\frac{\mu(D)}{D} \ll d(B_Q)Q^{-1}, \quad \text{and} \quad \prod_p(1-p^{-2}) = \frac{6}{\pi^2},$$

we finally obtain

$$\#\mathscr{F}_{B,Q}([0,1]) = \frac{Q^2}{2}\sum_{\substack{D|B_Q\\ D\le Q}}\frac{\mu(D)}{D}\left(\prod_{p\nmid B_Q}(1-p^{-2}) + O(D/Q)\right) + O(d(B_Q)Q\log Q)\,.$$

This yields the desired result after simplification. □

Lemma 2.2 *For any continuously differentiable function G with* supp $G\subset[0,1]$*, define*

$$S_{G,Q} = \sum_{\gamma\in\mathscr{F}_{B,Q}([0,1])} G(\gamma).$$

Then

$$S_{G,Q} = \frac{3Q^2\int_0^1 G(x)\mathrm{d}x}{\pi^2\delta(B_Q)} + O\left(\left(d(B_Q)\left|\int_0^1 G(x)\mathrm{d}x\right| + \|DG\|_\infty\right)Q\log Q\right),$$

where

$$||DG||_\infty = \sup_{x\in\mathbb{R}}|G'(x)|\,.$$

Proof. For $y \in (0,1)$, let $G(y) = \sum_{n\in\mathbb{Z}} a_n e(ny)$ be the Fourier series expansion of G. ,Using the identity (1.4), we have

$$\begin{aligned} S_{Q,G} &= \sum_{\gamma\in\mathscr{F}_{B,Q}([0,1])} \sum_{n\in\mathbb{Z}} a_n e\left(n(\gamma)\right) = \sum_{n\in\mathbb{Z}} a_n \sum_{\gamma\in\mathscr{F}_{B,Q}([0,1])} e(n\gamma) \\ &= \sum_{n\in\mathbb{Z}} a_n \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1\\ q\mid n}} q. \end{aligned}$$

That is

$$S_{Q,G} = \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1}} q \sum_{n\in\mathbb{Z}} a_{qn}.$$

The Fourier coefficients a_n are given by

$$a_n = \int_0^1 G(y)e(-ny)\mathrm{d}y = \int_{\mathbb{R}} G(y)e(-ny)\mathrm{d}y = \widehat{G}(n),$$

where $\widehat{G}$ is the Fourier transform of G defined by

$$\widehat{G}(x) = \int_{\mathbb{R}} G(y)e(-xy)\mathrm{d}y, \quad x\in\mathbb{R}.$$

Consider for each $y>0$ the function

$$G_y(x) = \frac{1}{y} G\left(\frac{x}{y}\right), \quad x\in\mathbb{R}.$$

By the properties of the Fourier transform, $\widehat{G}_y(x) = \widehat{G}(yx)$, and by Poisson summation (pp. 538, [16]),

$$\sum_{n\in\mathbb{Z}} a_{qn} = \sum_{n\in\mathbb{Z}} \widehat{G}(qn) = \sum_{n\in\mathbb{Z}} \widehat{G}_q(n) = \sum_{n\in\mathbb{Z}} G_q(n).$$

Hence

$$S_{Q,G} = \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1}} q \sum_{n\in\mathbb{Z}} \frac{1}{q} G\left(\frac{n}{q}\right).$$

Since Supp $G\subset[0,1]$ and G is continuously differentiable, by using simple Riemann integration ([14]) we obtain

$$\sum_{n\in\mathbb{Z}} \frac{1}{q} G\left(\frac{n}{q}\right) = \int_0^1 G(x)\mathrm{d}x + O\left(\frac{\|DG\|_\infty}{q}\right).$$

Therefore

$$S_{Q,G} = \left(\int_0^1 G(x)\mathrm{d}x\right) \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1}} q + O\left(\|DG\|_\infty Q\log Q\right).$$

Notice that

$$\#\mathscr{F}_{B,Q}([0,1]) = \sum_{\substack{d\le Q\\ \gcd(d,B_Q)=1}} \mu(d) \sum_{\substack{q\le Q/d\\ \gcd(q,B_Q)=1}} q.$$

Applying Lemma 2.1 we obtain the desired result. □

Lemma 2.3 *For any subinterval* $\mathbf{I} \subset [0,1]$, *denote by* $|\mathbf{I}|$ *the length of* $\mathbf{I}$. *Then*

$$\#\mathscr{F}_{B,Q}(\mathbf{I}) = |\mathbf{I}| \frac{3Q^2}{\pi^2 \delta(B_Q)} + O\left(\left(d(B_Q) + Q^{1/2} \right) Q \log Q \right),$$

as $Q \to \infty$.

P r o o f. We will approximate the characteristic function $\chi_{\mathbf{I}}$ of $\mathbf{I}$ by a C^1 function. To this end, consider the function $f(x) = 3x^2 - 2x^3$ for $x \in [0,1]$. First note the following properties:

- $f'(x) = 6x(1-x) \geq 0$ and $|f'(x)| \leq 3/2$ for $x \in [0,1]$;
- $f'(0) = f'(1) = 0, f(0) = 0, f(1) = 1$;
- $\int_0^1 f(x)\mathrm{d}x = 1/2$.

For real numbers $a < b < c < d$, define the function $g_{a,b,c,d} : \mathbb{R} \longrightarrow [0,1]$ by

$$g_{a,b,c,d}(t) = \begin{cases} 0 & : \quad t \leq a; \\ f\left(\frac{t-a}{b-a}\right) & : \quad a < t \leq b; \\ 1 & : \quad b < t \leq c; \\ f\left(1 - \frac{t-c}{d-c}\right) & : \quad c < t \leq d; \\ 0 & : \quad d < t. \end{cases}$$

It is easy to see that $g_{a,b,c,d} \in C^1(\mathbb{R})$ with Supp $g_{a,b,c,d} \subset [a,d]$, and

$$||Dg_{a,b,c,d}||_\infty \leq \frac{3}{2} \max\left\{ \frac{1}{b-a}, \frac{1}{d-c} \right\},$$

$$\int_{\mathbb{R}} g_{a,b,c,d}(x)\mathrm{d}x = c - b + \frac{b-a}{2} + \frac{d-c}{2}.$$

Now let $G = \chi_{[a,b]}$, the characteristic function of interval $\mathbf{I} = [a,b] \subset [0,1]$. Put $a_1 = a - \epsilon, a_2 = a + \epsilon, b_1 = b + \epsilon, b_2 = b - \epsilon$ and $G_1 = g_{a_1,a,b,b_1}, G_2 = g_{a,a_2,b_2,b}$, and denote

$$S_{Q,G_1} = \sum_{\gamma \in \mathscr{F}_{B \cdot Q}([0,1])} G_1(\gamma), \quad S_{Q,G_2} = \sum_{\gamma \in \mathscr{F}_{B \cdot Q}([0,1])} G_2(\gamma).$$

Since $G_2 \leq G = \chi_{[a,b]} \leq G_1$, it is clear that

$$S_{Q,G_2} \leq \#\mathscr{F}_{B,Q}(\mathbf{I}) \leq S_{Q,G_1}. \tag{2.1}$$

Noticing that

$$\int_{\mathbb{R}} G_1(x)\mathrm{d}x = b - a + \epsilon = |\mathbf{I}| + \epsilon, \quad \int_{\mathbb{R}} G_2(x)\mathrm{d}x = b - a - \epsilon = |\mathbf{I}| - \epsilon,$$

and

$$||DG_1||_\infty \leq \frac{3}{2\epsilon}, \quad ||DG_2||_\infty \leq \frac{3}{2\epsilon},$$

from Lemma 2.2 we obtain

$$S_{Q,G_1} \quad = \quad (|\mathbf{I}| + \epsilon) \frac{3Q^2}{\pi^2 \delta(B_Q)} + E_{Q,G_1},$$

$$S_{Q,G_2} \quad = \quad (|\mathbf{I}| - \epsilon) \frac{3Q^2}{\pi^2 \delta(B_Q)} + E_{Q,G_2},$$

where

$$E_{Q,G_1} = E_{Q,G_2} \ll \left(d(B_Q) + \frac{1}{\epsilon} \right) Q \log Q.$$

Choosing

$$\epsilon = Q^{-1/2},$$

and using the inequality (2.1), we get the desired result. □

3 Proof of Theorem 1.1

Assume now that $\omega(B_Q) \leq \frac{\log Q}{5 \log 2} + O(1)$, and $\mathbf{I} \subset [0,1]$ is a subinterval. Since B_Q is square free, $d(B_Q) = 2^{\omega(B_Q)} \ll Q^{1/5}$, by using Lemma 2.3 one obtains

$$N = \frac{\#\mathscr{F}_{B,Q}(\mathbf{I})}{|\mathbf{I}|} = \frac{3Q^2}{\pi^2 \delta(B_Q)} + O\left(Q^{3/2} \log Q \right). \tag{3.1}$$

Here the implied constant may depend on $|\mathbf{I}|$. One can also compute, by using Mertens' estimate ([16]) that

$$\delta(B_Q) = \prod_{p | B_Q} (1 + p^{-1}) \leq \exp \left(\sum_{p \leq \log Q} p^{-1} \right) \ll \log \log Q.$$

Our objective is to estimate, for any positive real number $\wedge$, the quantity

$$S_{B_Q}(\wedge) := \# \left\{ (x,y) \in \mathscr{F}_{B,Q}(\mathbf{I})^2 : x \neq y, x - y \in \frac{(0,\wedge)}{N} + \mathbb{Z} \right\},$$

as $Q \to \infty$. Let H, G be any two continuously differentiable functions with Supp $G \subset [0,1]$ and Supp $H \subset (0, \wedge)$ for some $\wedge > 0$. From now on, for simplicity, all the constants implied by the big "O" or "$\ll$" notation may depend on the functions H and G. Define

$$h(y) = \sum_{n \in \mathbb{Z}} H(N(y+n)), \quad y \in \mathbb{R},$$

and

$$S_{B_Q,H,G} = \sum_{\gamma, \gamma' \in \mathscr{F}_{B,Q}([0,1])} h(\gamma - \gamma') G(\gamma) G(\gamma').$$

We will estimate $S_{B_Q,H,G}$ instead of $S_{B_Q}(\wedge)$. Let $h(y) = \sum_{n \in \mathbb{Z}} c_n e(ny), G(y) = \sum_{n \in \mathbb{Z}} a_n e(ny)$ be the Fourier series expansions of h and G. Then we have

$$\begin{aligned}
S_{B_Q,H,G} &= \sum_{\gamma, \gamma' \in \mathscr{F}_{B,Q}([0,1])} \sum_m c_m e(m(\gamma - \gamma')) \sum_n a_n e(n\gamma) \sum_r a_r e(r\gamma') \\
&= \sum_{m,n,r} c_m a_n a_r \sum_{\gamma \in \mathscr{F}_{B,Q}([0,1])} e((m+n)\gamma) \sum_{\gamma' \in \mathscr{F}_{B,Q}([0,1])} e((r-m)\gamma') \\
&= \sum_{m,n,r} c_m a_n a_r \left(\sum_{\substack{d \leq Q \\ \gcd(d, B_Q) = 1}} \mu(d) \sum_{\substack{q \leq Q/d \\ \gcd(q, B_Q) = 1 \\ q | m+n}} q \right) \left(\sum_{\substack{d \leq Q \\ \gcd(d, B_Q) = 1}} \mu(d) \sum_{\substack{q \leq Q/d \\ \gcd(q, B_Q) = 1 \\ q | r-m}} q \right).
\end{aligned}$$

Changing the summation index as $m+n=m', r-m=n', m=r'$, hence $m=r', n=m'-r', r=n'+r'$, and one obtains

$$S_{B_Q,H,G} = \sum_{\substack{d_1,d_2\le Q\\ \gcd(d_1d_2,B_Q)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{q_1\le Q/d_1\\ q_2\le Q/d_2\\ \gcd(q_1q_2,B_Q)=1}} q_1q_2 \sum_{r\in\mathbb{Z}} c_r \sum_{m\in\mathbb{Z}} a_{q_1m-r} \sum_{n\in\mathbb{Z}} a_{q_2n+r}.$$

Consider for each $q>0$ and real number r the function

$$G_{r,q}(x) = \frac{1}{q} G\left(\frac{x}{q}\right) e\left(\frac{rx}{q}\right), \quad x\in\mathbb{R}.$$

Using the Fourier transform and an appropriate change of variable we obtain that for any $m\in\mathbb{Z}$,

$$\widehat{G}_{r,q}(m) \quad = \quad \widehat{G}(mq-r) = a_{qm-r}.$$

Applying the Poisson summation formula (pp. 538, [16]) one has

$$\sum_{m\in\mathbb{Z}} a_{q_1m-r} = \sum_{m\in\mathbb{Z}} \widehat{G}_{r,q_1}(m) = \sum_{m\in\mathbb{Z}} G_{r,q_1}(m) = \sum_{m\in\mathbb{Z}} \frac{1}{q_1} G\left(\frac{m}{q_1}\right) e\left(\frac{rm}{q_1}\right),$$

and similarly

$$\sum_{n\in\mathbb{Z}} a_{q_2n+r} = \sum_{n\in\mathbb{Z}} \frac{1}{q_2} G\left(\frac{n}{q_2}\right) e\left(\frac{-rn}{q_2}\right).$$

It follows that

$$\begin{aligned} &\sum_{r\in\mathbb{Z}} c_r \sum_{m\in\mathbb{Z}} a_{q_1m-r} \sum_{n\in\mathbb{Z}} a_{q_2n+r} \\ = \quad &\frac{1}{q_1q_2} \sum_{m,n} G\left(\frac{m}{q_1}\right) G\left(\frac{n}{q_2}\right) \sum_r c_r e\left(\left(\frac{m}{q_1} - \frac{n}{q_2}\right) r\right). \end{aligned}$$

The Fourier expansion of $h(y)$ gives us

$$\sum_r c_r e\left(\left(\frac{m}{q_1} - \frac{n}{q_2}\right) r\right) = h\left(\frac{m}{q_1} - \frac{n}{q_2}\right) = \sum_r H\left(N\left(r + \frac{m}{q_1} - \frac{n}{q_2}\right)\right).$$

Using this in the above formula for $S_{B_Q,H,G}$ and changing the symbols d's to r's and q's to d', we deduce that

$$\begin{aligned} S_{B_Q,H,G} \quad = \quad &\sum_{\substack{r_1,r_2\le Q\\ \gcd(r_1r_2,B_Q)=1}} \mu(r_1)\mu(r_2) \sum_{\substack{d_1\le Q/r_1, d_2\le Q/r_2\\ \gcd(d_1d_2,B_Q)=1}} \times \\ &\sum_{m,n,r\in\mathbb{Z}} G\left(\frac{m}{d_1}\right) G\left(\frac{n}{d_2}\right) H\left(N\left(r + \frac{m}{d_1} - \frac{n}{d_2}\right)\right). \end{aligned}$$

Since supp $G\subset[0,1]$, Supp $H\subset(0,\wedge)$, $d_1,d_2\le Q$ and $N\gg \frac{Q^2}{\log Q}$, if $r\ne 0$, then $H\left(N\left(r+\frac{m}{d_1}-\frac{n}{d_2}\right)\right)=0$ when Q is sufficiently large. For positive integers d_1,d_2, let

$$\delta=(d_1,d_2), \quad d_1=q_1\delta, \quad d_2=q_2\delta,$$

Clearly $(q_1,q_2)=1$, and there is a unique integer $\bar{q}_2$ such that $0<\bar{q}_2<q_1, \bar{q}_2q_2\equiv 1 \pmod{q_1}$. Take $\widetilde{a}_1=(1-\bar{q}_2q_2)/q_1$, so that $\widetilde{a}_1q_1+\bar{q}_2q_2=1$. Changing the summation index as

$$m'=q_2m-q_1n, \quad n'=\widetilde{a}_1m+\bar{q}_2n,$$

we have

$$m = \bar{q}_2 m' + q_1 n', \quad n = -\widetilde{a}_1 m' + q_2 n'.$$

Hence, by taking $r = 0$, we obtain that

$$S_{B_Q,H,G} = \sum_{\substack{r_1,r_2 \le Q \\ \gcd(r_1 r_2, B_Q)=1}} \mu(r_1)\mu(r_2) \sum_{\substack{q_1 \le Q/\delta r_1 \\ q_2 \le Q/\delta r_2 \\ \gcd(q_1,q_2)=1 \\ \gcd(q_1 q_2, B_Q)=1}} \sum_{m,n\in\mathbb{Z}} \times$$
$$G\left(\frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n\right)\right) G\left(\frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n - \frac{m}{q_1 q_2}\right)\right) H\left(\frac{Nm}{q_1 q_2 \delta}\right).$$

Taking into account the formula (3.1) and the fact that supp $H \subset (0, \wedge)$, when Q is large enough, to get a non-zero contribution from H, one must have

$$0 < \frac{Nm}{q_1 q_2 \delta} < \wedge,$$

which implies

$$0 < m\delta r_1 r_2 < 2C_\wedge,$$

where

$$C_\wedge = \frac{\pi^2 \delta(B_Q)\wedge}{3} \ll \log\log Q. \tag{3.2}$$

To get a non-zero contribution from G with supp $G \subset [0,1]$, one needs

$$0 \le \frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n\right) \le 1.$$

It follows that

$$-2C_\wedge < -m \le n \le \delta < 2C_\wedge.$$

Denoting by $\mathcal{A}_m$ the finite set of all such integers n for a fixed m, one has

$$\#\mathcal{A}_m \ll \log\log Q.$$

Using the above estimate and noticing that

$$G\left(\frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n - \frac{m}{q_1 q_2}\right)\right) = G\left(\frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n\right)\right) + O\left(\frac{m}{\delta q_1 q_2}\right),$$

and

$$H\left(\frac{Nm}{\delta q_1 q_2}\right) = H\left(\frac{3Q^2 m}{\pi^2 \delta(B_Q)\delta q_1 q_2}\right) + O\left(\frac{m}{\delta q_1 q_2} Q^{3/2}\log Q\right),$$

one can further simplify $S_{B_Q,H,G}$ as,

$$S_{B_Q,H,G} = \sum_{\substack{m r_1 r_2 \delta \le 2C_\wedge \\ n\in\mathcal{A}_m \\ \gcd(r_1 r_2, B_Q)=1}} \mu(r_1)\mu(r_2) \sum_{\substack{q_1 \le Q/\delta r_1 \\ q_2 \le Q/\delta r_2 \\ \gcd(q_1,q_2)=1 \\ \gcd(q_1 q_2, B_Q)=1}} G\left(\frac{1}{\delta}\left(\frac{\bar{q}_2 m}{q_1} + n\right)\right)^2 \times$$
$$H\left(\frac{3Q^2 m}{\pi^2 \delta(B_Q)\delta q_1 q_2}\right) + O(Q^{3/2}(\log Q)^4).$$

Next, for fixed m, r_1, r_2, δ, n, let us define

$$f(x) := G\left(\frac{1}{\delta}(mx+n)\right)^2, \quad h(x,y) := H\left(\frac{3Q^2m}{\pi^2\delta(B_Q)\delta xy}\right). \tag{3.3}$$

Then the inner sum of the main term of $S_{B_Q,H,G}$ on q_1, q_2, which will be treated below, can be written as

$$\sum = \sum_{\substack{q_1 \le Q/\delta r_1 \\ q_2 \le Q/\delta r_2 \\ \gcd(q_1,q_2)=1 \\ \gcd(q_1q_2,B_Q)=1}} f\left(\frac{\bar{q}_2}{q_1}\right) h(q_1,q_2), \tag{3.4}$$

where $\bar{q}_2$ is the unique integer such that $0 < \bar{q}_2 < q_1, \bar{q}_2 q_2 \equiv 1 \pmod{q_1}$. We will need several additional estimates. First of all,

$$\|f\|_\infty = \max_{x\in\mathbb{R}} |f(x)| = O(1), \quad \|Df\|_\infty = O(\log\log Q).$$

Next, since Supp $H \subset (0, \wedge)$, for $0 < x \le Q/\delta r_1, 0 < y \le Q/\delta r_2$, if $h(x,y) \ne 0$, then one must have $0 < \frac{3Q^2m}{\pi^2\delta(B_Q)\delta xy} < \wedge$, which implies

$$\frac{Q}{\delta r_1} \ge x > \frac{3Q^2m}{\wedge\pi^2\delta(B_Q)\delta y} \ge \frac{mr_1r_2\delta Q}{C_\wedge r_1\delta},$$

and similarly

$$\frac{Q}{\delta r_2} \ge y > \frac{mr_1r_2\delta Q}{C_\wedge r_2\delta}.$$

Here

$$\|h\|_\infty = O(1),$$

and

$$\left|\frac{\partial h}{\partial x}(x,y)\right| = \left|H'\left(\frac{3Q^2m}{\pi^2\delta(B_Q)\delta xy}\right)\right| \frac{3Q^2m}{\pi^2\delta(B_Q)\delta xy}\cdot\frac{1}{x} \ll \wedge\frac{C_\wedge r_1\delta}{mr_1r_2\delta Q} \ll \frac{\log Q}{Q}.$$

A similar inequality holds for $\frac{\partial h}{\partial y}(x,y)$ and hence

$$||Dh||_\infty \ll \frac{\log Q}{Q}.$$

Let K be a large positive integer which will be chosen later. Then (3.4) can be written as

$$\begin{aligned}\sum &= \sum_{i=0}^{K-1} \sum_{\substack{q_1 \le Q/\delta r_1 \\ q_2 \le Q/\delta r_2 \\ \gcd(q_1,q_2)=1 \\ \gcd(q_1q_2,B_Q)=1 \\ \bar{q}_2 \in [\frac{i}{K}q_1, \frac{i+1}{K}q_1)}} f\left(\frac{\bar{q}_2}{q_1}\right) h(q_1,q_2) \\ &= \sum_{i=0}^{K-1} f\left(\frac{i}{K}\right) \sum_{\substack{q_1 \le Q/\delta r_1 \\ q_2 \le Q/\delta r_2 \\ \gcd(q_1,q_2)=1 \\ \gcd(q_1q_2,B_Q)=1 \\ \bar{q}_2 \in [\frac{i}{K}q_1, \frac{i+1}{K}q_1)}} h(q_1,q_2) + O\left(\frac{Q^2\log\log Q}{K}\right).\end{aligned}$$

We need the following variations of results from [1]. Recall that for each region Ω in $\mathbb{R}^2$ and each C^1 function $F:\Omega \longrightarrow \mathbb{C}$, we denote

$$||F||_\infty = \sup_{(x,y)\in\Omega} |F(x,y)|,$$

and

$$||DF||_\infty = \sup_{(x,y)\in\Omega} \left(\left|\frac{\partial F}{\partial x}(x,y)\right| + \left|\frac{\partial F}{\partial y}(x,y)\right| \right).$$

For any subinterval $\mathbf{I} = [\alpha,\beta]$ of $[0,1]$, denote $\mathbf{I}_a = [(1-\beta)a, (1-\alpha)a]$.

Lemma 3.1 *Let $\Omega \subset [1,R]\times[1,R]$ be a convex region and let F be a C^1 function on Ω. For any square-free integer A and any subinterval $\mathbf{I}\subset[0,1]$ one has*

$$\sum_{\substack{(a,b)\in\Omega\bigcap\mathbb{Z}^2\\ \gcd(a,b)=1\\ \bar{b}\in\mathbf{I}_a\\ \gcd(ab,A)=1}} F(a,b) = \frac{6|\mathbf{I}|}{\pi^2}\frac{\delta'(A)}{\delta(A)^2}\iint_\Omega F(x,y)\,\mathrm{d}x\mathrm{d}y + E_{R,\Omega,F,\mathbf{I},A},$$

where

$$E_{R,\Omega,F,\mathbf{I},A} \ll_\epsilon \quad m_F\|F\|_\infty R^{3/2+\epsilon}d(A) + \|F\|_\infty d(A)^2 R\log R + \|Df\|_\infty \mathit{Area}(\Omega)(\log R)d(A)\delta(A),$$

for any $\epsilon>0$, where $\bar{b}$ denotes the multiplicative inverse of $b \pmod a$, i.e., $1\le\bar{b}\le a, b\bar{b}\equiv 1 \pmod a$, m_F is an upper bound for the number of intervals of monotonicity of each of the functions $y\mapsto F(x,y)$, and the function δ', δ are defined in (1.1).

The proof of Lemma 3.1 goes along the same line as that of Lemma 8 in [1], where Weil type estimates ([9, 19]) for certain weighted incomplete Kloosterman sums play a crucial role.

Using the above lemma, one has

$$\sum_{\substack{q_1\le Q/\delta r_1\\ q_2\le Q/\delta r_2\\ \gcd(q_1,q_2)=1\\ \gcd(q_1q_2,B_Q)=1\\ \bar{q}_2\in[\frac{i}{K}q_1,\frac{i+1}{K}q_1)}} h(q_1,q_2) = \frac{6\delta'(B_Q)}{K\pi^2\delta(B_Q)^2}\int_0^{\frac{Q}{\delta r_1}}\int_0^{\frac{Q}{\delta r_2}} h(x,y)\mathrm{d}x\mathrm{d}y + O_\epsilon\left(Q^{\frac{3}{2}+\epsilon}d(B_Q)\right),$$

which is

$$\frac{6\delta'(B_Q)}{\pi^2\delta(B_Q)^2}\iint_{[0,1]^2} H\left(\frac{3m\delta r_1r_2}{\pi^2\delta(B_Q)xy}\right)\mathrm{d}x\mathrm{d}y + O_\epsilon\left(Q^{\frac{3}{2}+\epsilon}d(B_Q)\right).$$

Returning to the sum $\sum$ we obtain that

$$\sum = \frac{6\delta'(B_Q)Q^2}{\pi^2\delta(B_Q)^2\delta^2r_1r_2}\int_0^1 f(x)\mathrm{d}x\iint_{[0,1]^2} H\left(\frac{3m\delta r_1r_2}{\pi^2\delta(B_Q)xy}\right)\mathrm{d}x\,\mathrm{d}y + O_\epsilon\left(KQ^{\frac{3}{2}+\epsilon}d(B_Q)\right) + O\left(\frac{Q^2\log\log Q}{K}\right).$$

We may choose

$$K = [Q^{\frac{1}{4}}],$$

and use $d(B_Q)\ll Q^{1/5}$, to obtain an error term $E_1 \ll_\epsilon Q^{2-\frac{1}{20}+\epsilon}$. It is easy to see from the definition (3.3) that

$$\sum_{n\in\mathbb{Z}}\int_0^1 f(x)\mathrm{d}x = \delta\cdot\int_0^1 G(z)^2\mathrm{d}z.$$

Using this and all the above results, we finally conclude that

$$S_{B_Q,H,G} = \frac{6\delta'(B_Q)Q^2}{\pi^2\delta(B_Q)^2}\left(\int_0^1 G(z)^2\mathrm{d}z\right)\sum_{\substack{1\le r_1r_2m\delta\le 2C_\wedge\\ \gcd(r_1r_2\delta,B_Q)=1}}\frac{\mu(r_1)\mu(r_2)}{\delta r_1r_2}\times$$
$$\iint_{[0,1]^2} H\left(\frac{3m\delta r_1r_2}{\pi^2\delta(B_Q)xy}\right)\mathrm{d}x\,\mathrm{d}y + O_\epsilon(Q^{2-\frac{1}{20}+\epsilon})\,.$$

4 Completion of the proof of Theorem 1.1

Denote the main term of $S_{B_Q,H,G}$ by $S'_{B_Q,H,G}$. If

$$\lim_{Q\to\infty}\delta(B_Q)=\delta(B)<\infty,$$

we follow the computation of $\mathcal{S}_2$ in [3]. Since Supp $H\subset(0,\wedge)$, we put $\lambda=\frac{3m\delta r_1r_2}{\pi^2\delta(B_Q)xy}$, then the double integral in the above $S_{B_Q,H,G}$ becomes

$$\int_{\frac{\delta mr_1r_2}{C_\wedge}}^1\int_{\frac{\delta mr_1r_2}{C_\wedge x}}^1 H\left(\frac{\wedge\delta mr_1r_2}{C_\wedge xy}\right)\mathrm{d}y\,\mathrm{d}x = \frac{\wedge\delta mr_1r_2}{C_\wedge}\int_{\frac{\delta mr_1r_2}{C_\wedge}}^1\int_{\frac{\wedge\delta mr_1r_2}{C_\wedge x}}^\wedge \frac{H(\lambda)}{x\lambda^2}\mathrm{d}\lambda\,\mathrm{d}x\,,$$

which is

$$\frac{\wedge\delta mr_1r_2}{C_\wedge}\int_{\frac{\wedge\delta mr_1r_2}{C_\wedge}}^\wedge\frac{H(\lambda)}{\lambda^2}\log\left(\frac{C_\wedge\lambda}{\wedge\delta mr_1r_2}\right)\mathrm{d}\lambda.$$

Inserting this back into $S_{B_Q,H,G}$, we obtain

$$S'_{B_Q,H,G} = \frac{18Q^2\delta'(B_Q)\left(\int_0^1 G(z)^2\mathrm{d}z\right)}{\pi^4\delta(B_Q)^3}\sum_{1\le k\le 2C_\wedge}\int_{\frac{3k}{\pi^2\delta(B_Q)}}^\wedge\frac{H(\lambda)}{\lambda^2}\log\left(\frac{\pi^2\lambda\delta(B_Q)}{3k}\right)\mathrm{d}\lambda\times$$
$$\sum_{\substack{m\delta r_1r_2=k\\ \gcd(\delta r_1r_2,B_Q)=1}}\mu(r_1)\mu(r_2)m\,.$$

Let $u=\gcd(k,B_Q)$, then

$$\sum_{\substack{m\delta r_1r_2=k\\ \gcd(\delta r_1r_2,B_Q)=1}}\mu(r_1)\mu(r_2)m = \sum_{\delta r_1r_2\mid\frac{k}{u}}\mu(r_1)\mu(r_2)\frac{k}{\delta r_1r_2}=k\sum_{r_1\mid\frac{k}{u}}\frac{\mu(r_1)}{r_1}\sum_{\delta\mid\frac{k}{ur_1}}\frac{1}{\delta}\sum_{r_2\mid\frac{k}{ur_1\delta}}\frac{\mu(r_2)}{r_2}$$
$$k\sum_{r_1\mid\frac{k}{u}}\frac{\mu(r_1)}{r_1}\sum_{\delta\mid\frac{k}{ur_1}}\frac{1}{\delta}\frac{\varphi(k/(ur_1\delta))}{k/(ur_1\delta)} = u\sum_{r_1\mid\frac{k}{u}}\mu(r_1)\sum_{\delta\mid\frac{k}{ur_1}}\varphi(k/(ur_1\delta))=u\sum_{r_1\mid\frac{k}{u}}\mu(r_1)\frac{k}{ur_1}=u\varphi(k/u)\,.$$

Also notice that

$$\frac{3k}{\pi^2\delta(B_Q)}\le\wedge\Longrightarrow k\le\frac{\pi^2\delta(B_Q)\wedge}{3}=C_\wedge,$$

$$\lim_{Q\to\infty}\delta(B_Q)=\delta(B)<\infty,\quad \lim_{Q\to\infty}\delta'(B_Q)=\delta'(B)<\infty,$$

and

$$\#\mathscr{F}_{B,Q}(\mathbf{I})=|\mathbf{I}|\frac{3Q^2}{\pi^2\delta(B_Q)}+O\left(Q^{3/2}\log Q\right).$$

One concludes that

$$\lim_{Q\to\infty}\frac{S_{B_Q,H,G}}{\#\mathscr{F}_{B,Q}(\mathbf{I})}=\frac{\int_0^1 G(z)^2\mathrm{d}z}{|\mathbf{I}|}\int_0^{\wedge}H(\lambda)g_B(\lambda)\mathrm{d}\lambda\,,$$

where the function $g_B(\lambda)$ is defined in Theorem 1.1. Now using standard approximation arguments one sees that

$$\lim_{Q\to\infty}\frac{S_{B_Q}(\wedge)}{\#\mathscr{F}_{B,Q}(\mathbf{I})}=\int_0^{\wedge}g_B(\lambda)\mathrm{d}\lambda\,,$$

and $g_B(\lambda)$ is the limiting pair correlation function of $\mathscr{F}_{B,Q}(\mathbf{I})$ as Q goes to infinity. This completes the proof of Theorem 1.1 for the case $\delta(B)<\infty$.

If $\delta(B)=\infty$, then $C_\wedge=\frac{\pi^2\delta(B_Q)\wedge}{3}\to\infty$. We focus on the sum

$$S''_{B_Q,H,G} = \sum_{\substack{1\le r_1r_2\delta\le 2C_\wedge\\ \gcd(r_1r_2\delta,B_Q)=1}}\frac{\mu(r_1)\mu(r_2)}{\delta r_1r_2}\sum_{m\le\frac{C_\wedge}{\delta r_1r_2}}\iint_{[0,1]^2}H\left(\frac{\wedge\delta r_1r_2m}{C_\wedge xy}\right)\mathrm{d}x\,\mathrm{d}y\,.$$

Let

$$L=\frac{C_\wedge}{\delta r_1r_2}\,,$$

and define a function f by

$$f(z)=\iint_{[0,1]^2}H\left(\frac{\wedge z}{xy}\right)\mathrm{d}x\mathrm{d}y,\quad z\in\mathbb{R}.$$

One can see that

$$S''_{B_Q,H,G} = \sum_{\substack{1\le r_1r_2\delta\le 2C_\wedge\\ \gcd(r_1r_2\delta,B_Q)=1}}\frac{\mu(r_1)\mu(r_2)}{\delta r_1r_2}\sum_{1\le m\le L}f\left(\frac{m}{L}\right),$$

and

$$\|f\|_\infty\ll 1,\quad \log L\le\log C_\wedge.$$

Since Supp $H\subset(0,\wedge)$, there is a non-zero contribution to the integral in the definition of f only if $0<\frac{\wedge z}{xy}<\wedge$, therefore z has to satisfy $1\ge xy>z>0$. It is clear that

$$f'(z)\ll\iint_{xy>z}\frac{1}{xy}\mathrm{d}x\mathrm{d}y\ll\frac{1}{z}.$$

Therefore,

$$\begin{aligned}\frac{1}{L}\sum_{1\le m\le L}f\left(\frac{m}{L}\right)-\int_{\mathbb{R}}f(z)\mathrm{d}z &= \sum_{m=2}^{[L]}\int_{(m-1)/L}^{m/L}f\left(\frac{m}{L}\right)-f(z)\mathrm{d}z+O\left(\frac{1}{L}\right)\\ &\ll \sum_{m=2}^{[L]}\int_{(m-1)/L}^{m/L}\frac{1}{L}\frac{L}{m}\mathrm{d}x+O\left(\frac{1}{L}\right)\ll\frac{\log L}{L}\,.\end{aligned}$$

Also notice that

$$\int_{\mathbb{R}}f(z)\mathrm{d}z=\iint_{[0,1]^2}\left(\int_{\mathbb{R}}H\left(\frac{\wedge z}{xy}\right)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y=\frac{1}{4\wedge}\int_{\mathbb{R}}H(x)\mathrm{d}x.$$

Using this and the above estimate one obtains

$$S''_{B_Q,H,G} = \sum_{\substack{1\le r_1r_2\delta\le 2C_\wedge \\ \gcd(r_1r_2\delta,B_Q)=1}} \frac{\mu(r_1)\mu(r_2)}{\delta r_1r_2}\left(\frac{C_\wedge}{4\wedge\delta r_1r_2}\int_{\mathbb{R}} H(x)\mathrm{d}x + O\left(\log C_\wedge\right)\right)$$
$$= \frac{C_\wedge}{4\wedge}\int_{\mathbb{R}} H(x)\mathrm{d}x \sum_{\substack{1\le r_1r_2\delta\le 2C_\wedge \\ \gcd(r_1r_2\delta,B_Q)=1}} \frac{\mu(r_1)\mu(r_2)}{\delta^2r_1^2r_2^2} + O((\log C_\wedge)^4)\,.$$

Since

$$\sum_{\substack{\delta\le \frac{2C_\wedge}{r_1r_2} \\ \gcd(\delta,B_Q)=1}} \delta^{-2} = \sum_{\gcd(\delta,B_Q)=1} \delta^{-2} + O\left(\frac{r_1r_2}{C_\wedge}\right) = \prod_{p\nmid B_Q}\left(1-p^{-2}\right)^{-1} + O\left(\frac{r_1r_2}{C_\wedge}\right),$$

$$\sum_{\substack{r_2\le \frac{2C_\wedge}{r_1} \\ \gcd(r_2,B_Q)=1}} \frac{\mu(r_2)}{r_2^2} = \sum_{\gcd(r_2,B_Q)=1} \frac{\mu(r_2)}{r_2^2} + O\left(\frac{r_1}{C_\wedge}\right) = \prod_{p\nmid B_Q}\left(1-p^{-2}\right) + O\left(\frac{r_1}{C_\wedge}\right),$$

and

$$\sum_{\substack{r_1\le 2C_\wedge \\ \gcd(r_1,B_Q)=1}} \frac{\mu(r_1)}{r_1^2} = \prod_{p\nmid B_Q}\left(1-p^{-2}\right) + O\left(\frac{1}{C_\wedge}\right),$$

one sees that

$$\sum_{\substack{1\le r_1r_2\delta\le 2C_\wedge \\ \gcd(r_1r_2\delta,B_Q)=1}} \frac{\mu(r_1)\mu(r_2)}{\delta^2r_1^2r_2^2} = \sum_{\substack{r_1\le C_\wedge \\ \gcd(r_1,B_Q)=1}} \frac{\mu(r_1)}{r_1^2} \sum_{\substack{r_2\le \frac{C_\wedge}{r_1} \\ \gcd(r_2,B_Q)=1}} \frac{\mu(r_2)}{r_2^2} \sum_{\substack{\delta\le \frac{C_\wedge}{r_1r_2} \\ \gcd(\delta,B_Q)=1}} \delta^{-2}$$
$$= \prod_{p\nmid B_Q}(1-p^{-2}) + O\left(\frac{(\log C_\wedge)^2}{C_\wedge}\right),$$

and

$$S''_{B_Q,H,G} = \frac{C_\wedge}{4\wedge}\prod_{p\nmid B_Q}(1-p^{-2})\int_{\mathbb{R}} H(x)\mathrm{d}x + O((\log C_\wedge)^4)\,.$$

Now returning to $S_{B_Q,H,G}$, using the definition of $C_\wedge$ in (3.2), we finally conclude that

$$S_{B_Q,H,G} = \frac{3Q^2\left(\int_{\mathbb{R}} G(z)^2\mathrm{d}z\right)\left(\int_{\mathbb{R}} H(x)\mathrm{d}x\right)}{\pi^2\delta(B_Q)} + O\left(\frac{Q^2(\log\delta(B_Q))^4}{\delta(B_Q)^2}\right)\,.$$

Since $\delta(B_Q)\to\infty$, one has

$$\lim_{Q\to\infty}\frac{S_{B_Q,H,G}}{\#\mathscr{F}_{B,Q}(\mathbf{I})} = \frac{\int_0^1 G(z)^2\mathrm{d}z}{|\mathbf{I}|}\int_0^\wedge H(x)\mathrm{d}x\,.$$

A standard argument with smooth functions as before completes the proof for the case $\delta(B)=\infty$. This finishes the proof of Theorem 1.1.

Acknowledgements The authors are grateful to the referee for many valuable suggestions.

References

[1] F. Boca, C. Cobeli, A. Zaharescu, *A conjecture of R. R. Hall on Farey points*, J. Reine Angew. Math. 535 (2001), 207–236.
[2] F. P. Boca, R. N. Gologan, A. Zaharescu, *The average length of a trajectory in a certain billiard in a flat two-torus*, New York J. Math. 9 (2003), 303–330.
[3] F. P. Boca, A. Zaharescu, *The Correlations of Farey Fractions*, J. London Math. Soc. (2) 72 (2005), 25–39.
[4] F.P. Boca, A. Zaharescu, *The distribution of the free path lengths in the periodic two-dimensional Lorentz gas in the small-scatterer limit*, Comm. Math. Phys. 269 (2007), no. 2, 425–471.
[5] F. J. Dyson, *Statistical theory of the energy levels of complex systems*, I. J. Mathematical Phys. 3 (1962), 140–156.
[6] F. J. Dyson, *Statistical theory of the energy levels of complex systems*, II. J. Mathematical Phys. 3(1962), 157165.
[7] F. J. Dyson, *Statistical theory of the energy levels of complex systems*, III. J. Mathematical Phys. 3(1962), 166175.
[8] H. M. Edwards, "Riemann's zeta function", Dover, Mineola, NY 2001.
[9] T. Esterman, *On Kloosterman's sums*, Mathematika 8 (1961), 83–86.
[10] J. Franel, *Les suites de Farey et le probl'eme des nombres premiers*, Göttinger Nachr. (1924), 198–201.
[11] D. A. Hejhal, *On the triple correlation of zeros of the zeta function*, Internat. Math. Res. Notices, 7 (1994).
[12] M. N. Huxley, A. Zhigljavsky, *On the distribution of Farey fractions and hyperbolic lattice points*, Period. Math. Hungar. 42 (2001), 191–198.
[13] N. Katz, P. Sarnak, *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 1, 1–26.
[14] L. Kuipers, H. Niederreiter, "Uniform distribution of sequences", Pure and Applied Mathematics, Wiley-Interscience, 1974.
[15] E. Landau, *Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel*, Göttinger Nachr. (1924), 202–206.
[16] H. L. Montgomery, R. Vaughan, Robert C, "Multiplicative number theory. I. Classical theory", Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.
[17] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol XXIV, St. Louis Univ., St. Louis, Mo., 1972), 181–193. Amer. Math. Soc., Providence, R.I., 1973.
[18] Z. Rudnick, P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. 81 (1996), no. 2, 269-322.
[19] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci.USA 34 (1948), 204–207.
[20] H. Weyl, *Über die Gleichverteilung von Zahlen mod*, Eins, Math. Ann. 77 (1916), no. 3, 313–352.
[21] E. Wigner, *Random matrices in physics*, SIAM Review 9 (1967), 123.