Theory of Ordinary Differential Equations

Existence, Uniqueness and Stability

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Contents

1 Existence and Uniqueness 1
- 1.1 Some Basics 1
- 1.2 Uniqueness Theorem 6
- 1.3 Continuity 8
- 1.4 Existence Theorem 11
- 1.5 Local Existence Theorem and The Peano Theorem 18
  - 1.5.1 Local Existence Theorem 18
  - 1.5.2 The Peasno Theorem 19
- 1.6 Linear Systems 22
- 1.7 Continuation of Solutions 25
- 1.8 Miscellaneous Problems 27

2 Plane Autonomous Systems 31
- 2.1 Plane Autonomous Systems 31
- 2.2 Linear Autonomous Systems 37
- 2.3 Complete Classification for Linear Autonomous Systems 41
- 2.4 Liapunov Direct Method 48
- 2.5 A Test For Instability 56
- 2.6 Nonlinear Oscillations 60
  - 2.6.1 Undamped Oscillations 60

2.6.2 Damped Oscillations *61*
2.7 Miscellaneous Problems *62*

1

Existence and Uniqueness

1.1 SOME BASICS

A normal system of first order ordinary differential equations (ODEs) is

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} & = & X_1(x_1, \ldots, x_n; t), \\ & \vdots & \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} & = & X_n(x_1, \ldots, x_n; t). \end{cases} \tag{1.1}$$

Many varieties of ODEs can be reduced to this form. For example, consider an n-th order ODE

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = F\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}\right).$$

Let $x_1 = x, x_2 = \dfrac{\mathrm{d}x}{\mathrm{d}t}, \ldots, x_n = \dfrac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}$. Then the ODE can be changed to

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} & = & x_2, \\ & \vdots & \\ \dfrac{\mathrm{d}x_{n-1}}{\mathrm{d}t} & = & x_n, \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} & = & F(t, x_1, \ldots, x_n). \end{cases}$$

Let us review some notations and facts on vectors and vector-valued functions. The normal system (1.1) can be written as its vector form:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t).$$

For a vector $\mathbf{x} = (x_1, \ldots, x_n)$, define $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$. The inner product is defined to be $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ where $\mathbf{y} = (y_1, \ldots, y_n)$. We have the triangle inequality

$$\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|,$$

and the Schwarz inequality

$$|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

For a vector valued function $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$, we have

$$\mathbf{x}'(t) = \left(x_1'(t), \ldots, x_n'(t)\right),$$

and

$$\int_a^b \mathbf{x}(t)\mathrm{d}t = \left(\int_a^b x_1(t)\mathrm{d}t, \ldots, \int_a^b x_n(t)\mathrm{d}t\right).$$

We have the following useful inequality

$$\left\|\int_a^b \mathbf{x}(t)\mathrm{d}t\right\| \le \int_a^b \|\mathbf{x}(t)\|\,\mathrm{d}t.$$

A vector field $\mathbf{X}(\mathbf{x}) = \big(X_1(x_1, \ldots, x_n), \ldots, X_n(x_1, \ldots, x_n)\big)$ is aid to be continuous if each X_i is a continuous function of $x_1, \ldots, x_n$.

Since only a few simple types of differential equations can be solved explicitly in terms of known elementary function, in this chapter, we are going to explore the conditions on the function $\mathbf{X}$ such that the differential system has a solution. We also study whether the solution is unique, subject some additional initial conditions.

Example 1.1.1 *Show that the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{2}y^{-1}$$

does not have solution satisfying $y(0) = 0$ *for* $t > 0$*.*

Solution Acturally, the general solution of this differential equation is

$$y^2 = -t + C,$$

where C is an arbitrary constant. The initial condition implies $C = 0$. Thus, we have $y^2 = -t$, that shows there exists no solution for $t > 0$. □

Example 1.1.2 *Show that the differential equation $x' = x^{2/3}$ has infinitely many solutions satisfying $x(0) = 0$ on every interval $[0, b]$.*

Solution Define

$$x_c(t) = \begin{cases} 0, & \text{if } 0 \le t < c; \\ \dfrac{(t-c)^3}{27}, & \text{if } c \le t \le b. \end{cases}$$

It is easy to check for any c, the function x_c satisfies the differential equation and $x_c(0) = 0$. □

Definition 1.1.1 *A vector-valued function $\mathbf{X}(\mathbf{x}, t)$ is said to satisfy a Lipschitz condition in a region $\mathcal{R}$ in $(\mathbf{x}, t)$-space if, for some constant L (called the Lipschitz constant), we have*

$$\|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\| \le L\|\mathbf{x} - \mathbf{y}\|, \tag{1.2}$$

whenever $(\mathbf{x}, t) \in \mathcal{R}$ and $(\mathbf{y}, t) \in \mathcal{R}$.[1]

Lemma 1.1.2 *If $\mathbf{X}(\mathbf{x}, t)$ has continuous partial derivatives on a bounded closed convex domain $\mathcal{D}$, then it satisfies a Lipschitz condition in $\mathcal{R}$.*[2]

Proof Denote

$$M = \sup_{\substack{\mathbf{x} \in \mathcal{R} \\ 1 \le i, j \le n}} \left|\partial X_i / \partial x_j\right|.$$

Since $\mathbf{X}$ has continuous partial derivatives on the bounded closed region $\mathcal{R}$, we know that M is finite. For each component X_i we have, for fixed $\mathbf{x}, \mathbf{y}, t$,

$$\frac{\mathrm{d}}{\mathrm{d}s} X_i(\mathbf{x} + s\mathbf{y}, t) = \sum_{k=1}^{n} \frac{\partial X_i}{\partial x_k}(\mathbf{x} + s\mathbf{y}, t) y_k.$$

By integrating both sides from 0 to 1 and using the mean value theorem, we have

$$X_i(\mathbf{x} + \mathbf{y}, t) - X_i(\mathbf{x}, t) = \sum_{k=1}^{n} \frac{\partial X_i}{\partial x_k}(\mathbf{x} + \sigma_i \mathbf{y}, t) y_k,$$

[1] It is worth to point out the following:

(i) The left hand side of the inequality (1.2) is evaluated at $(\mathbf{x}, t)$ and $(\mathbf{y}, t)$, with the same t for these two points.

(ii) The constant L is independent of $\mathbf{x}$, $\mathbf{y}$ and t. However, it depends on $\mathcal{R}$. In other words, for a given function $\mathbf{X}(\mathbf{x}, t)$, its Lipschitz constant may change if the domain $\mathcal{R}$ is different. In fact, for the same function $\mathbf{X}(\mathbf{x}, t)$, it can be a Lipschitz function in some regions, but not a Lipschitz function in some other regions.

[2] Here are some explanation of some concepts and terminologies in the Lemma.

(i) A bounded closed domain is also called compact. It has the following property that any continuous function on a compact set has an absolute maximum and absolute minimum.

(ii) That $\mathbf{X}(\mathbf{x}, t) = \Big(X_1(x_1, \ldots, x_n, t), \ldots, X_n(x_1, \ldots, x_n, t)\Big)$ has continuous partial derivatives means that $\dfrac{\partial X_i}{\partial x_j}$ and $\dfrac{\partial X_i}{\partial t}$ are continuous for all i, j. Sometimes we use $\mathbf{X}(\mathbf{x}, t) \in C^1(\mathcal{D})$ for this.

(iii) A convex domain $\mathcal{D}$ means that for any two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{D}$, $(1-t)\mathbf{x} + t\mathbf{y} \in \mathcal{D}$ for $0 \le t \le 1$.

for some σ_i between 0 and 1. The assumption that $\mathcal{R}$ is convex implies that $(\mathbf{x}+\sigma_i\mathbf{y},t)\in\mathcal{R}$. The Schwarz inequality gives

$$\begin{aligned}\left|\sum_{k=1}^{n}\frac{\partial X_i}{\partial x_k}(\mathbf{x}+\sigma_i\mathbf{y},t)y_k\right| &\le \left(\sum_{k=1}^{n}\left|\frac{\partial X_i}{\partial x_k}(\mathbf{x}+\sigma_i\mathbf{y},t)\right|^2\right)^{1/2}\left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\\ &\le \left(\sum_{k=1}^{n}M^2\right)^{1/2}\cdot\|\mathbf{y}\| = \sqrt{n}M\cdot\|\mathbf{y}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathbf{X}(\mathbf{x}+\mathbf{y},t)-\mathbf{X}(\mathbf{x},t)\| &= \left(\sum_{i=1}^{n}|X_i(\mathbf{x}+\mathbf{y},t)-X_i(\mathbf{x},t)|^2\right)^{1/2}\\ &\le \left(\sum_{i=1}^{n}\left(\sqrt{n}M\cdot\|\mathbf{y}\|\right)^2\right)^{1/2} = nM\cdot\|\mathbf{y}\|.\end{aligned}$$

The Lipschitz condition follows, with the Lipschitz constant nM. □

Example 1.1.3 *Determine whether the function*

$$X(x,t)=\frac{x^2+1}{x}\cdot t$$

satisfies a Lipschitz condition in the domains: (1) $\mathcal{R}_1=[1,2]\times[0,1]$*;* (2) $\mathcal{R}_2=(1,2)\times[0,1]$*;* (3) $\mathcal{R}_3=[1,2]\times[0,+\infty)$*;* (4) $\mathcal{R}_4=[1,+\infty)\times[0,T]$*;* (5) $\mathcal{R}_5=(0,1)\times[0,1]$.

Solution (1) Since the function $X(x,t)$ is continuously differentiable in the bounded closed convex domain $\mathcal{R}=[1,2]\times[0,1]$, by Lemma 1.1.2, we know that the function satisfies a Lipschitz conditio in this region.

(2) Since the function $X(x,t)$ satisfis a Lipschitz condition in $\mathcal{R}_1$, and $\mathcal{R}_2\subset\mathcal{R}_1$, we know that the function satisfis the same Lipschitz inequality in $\mathcal{R}_2$. Hence the function $X(x,t)$ satisfis a Lipschitz condition in $\mathcal{R}_2$.

(3) Since $\mathcal{R}_3=[1,2]\times[0,+\infty)$ is not a bounded region, we cannot apply Lemma 1.1.2 in this case. Since

$$\frac{|X(x,t)-X(y,t)|}{|x-y|}=\left|\frac{xy+1}{xy}\right|\cdot|t|>|t|\to\infty,$$

as $t\to+\infty$, there exists no contant L, independent of x,y and t, such that

$$|X(x,t)-X(y,t)|\le L\,|x-y|\,.$$

Hence, the function X is not a Lipschitz function in $\mathcal{R}_3$.

(4) Again, $\mathcal{R}_4=[1,+\infty)\times[0,T]$ is not bounded and Lemma 1.1.2 does not apply in this case. For $(x,t),(y,t)\in\mathcal{R}_4$,

$$\begin{aligned}|X(x,t)-X(y,t)| &= \left|\frac{xy+1}{xy}\right|\cdot|t|\cdot|x-y|\\ &\le \left(1+\frac{1}{xy}\right)\cdot T\cdot|x-y|\\ &\le 2T|x-y|.\end{aligned}$$

Thus, the function X is a Lipschitz function in $\mathcal{R}_4$, with $L=2T$.

(5) Since $\mathcal{R}_5 = (0,1) \times [0,1]$ is not a closed region, Lemma 1.1.2 does not apply in this case. In fact, the function X is continuously differentiable in $\mathcal{R}$, with

$$X'(x) = \frac{x^2 - 1}{x^2}.$$

But the derivative is not bounded, since $X'(x) \to \infty$ as $x \to 0+$.

Actually, we can show that the function X does not satisfy any Lipschitz condition in $\mathcal{R}_5$. To see that, we take $x = \frac{1}{n}$, $y = \frac{2}{n}$ and $t = 1$, and have

$$|X(x,t) - X(y,t)| = \frac{n}{2} - \frac{1}{n},$$

for $n \geq 2$. Obviously, there exists no constant L such that $\frac{n}{2} - \frac{1}{n} \leq L \cdot \frac{1}{n} = L|x - y|$. Therefore, the function X does not satisfy any Lipschitz conditions in the domain $\mathcal{R}_5$. □

Example 1.1.4 *Reduce the ODE* $\frac{d^3x}{dt^3} + x^2 = 1$ *to an equivalent first-order system and determine in which domain or domains the resulting system satisfies a Lipschitz condition.*

Solution Let $x_1 = x$, $x_2 = x'$, $x_3 = x''$, we get the equivalent first order system

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} & = & x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} & = & x_3, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} & = & 1 - x_1^2. \end{cases}$$

Denote $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{X}(\mathbf{x}, t) = (x_2, x_3, 1 - x_1^2)$. Then, $\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t) = (x_2 - y_2, x_3 - y_3, -x_1^2 + y_1^2)$, which implies

$$\|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\| = \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_1 - y_1)^2 (x_1 + y_1)^2}.$$

We can see that whenever x_1 and y_1 are bounded, to be precise, $x_1 + y_1$ is bounded, the Lipschitz condition holds. In fact, consider the domain

$$\mathcal{R} = \left\{ (x_1, x_2, x_3, t) \;\middle|\; |x_1| \leq M \right\}$$

for some constant M. Take $L = \sqrt{\max\{4M^2, 1\}}$, then whenever $(\mathbf{x}, t), (\mathbf{y}, t) \in \mathcal{R}$, we have

$$\begin{aligned} \|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\| &\leq \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2 + 4M^2 (x_1 - y_1)^2} \\ &\leq \sqrt{L^2 (x_2 - y_2)^2 + L^2 (x_3 - y_3)^2 + L^2 (x_1 - y_1)^2} = L \|\mathbf{x} - \mathbf{y}\|. \end{aligned}$$

Therefore $\mathbf{X}(\mathbf{x}, t)$ satisfies the Lipschitz condition with the Lipschitz constant $L = \sqrt{\max\{4M^2, 1\}}$ in the domain $\mathcal{R}$. □

1.2 UNIQUENESS THEOREM

In this section, we prove the following uniqueness theorem.

Theorem 1.2.1 (**Uniqueness Theorem**) *If the vector field* $\mathbf{X}(\mathbf{x}, t)$ *satisfies a Lipschitz condition in a domain* $\mathcal{R}$*, then there is at most one solution* $\mathbf{x}(t)$ *of the differential system*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t), \tag{1.3}$$

that satisfies a given initial condition $\mathbf{x}(a) = \mathbf{c}$ *in* $\mathcal{R}$*.*

Proof Assume that $\mathbf{X}(\mathbf{x}, t)$ satisfies the Lipschitz condition

$$\|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\| \leq L \|\mathbf{x} - \mathbf{y}\|,$$

for any $(\mathbf{x}, t), (\mathbf{y}, t) \in \mathcal{R}$. Let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ and $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$ be two solutions satisfying the same initial condition $\mathbf{x}(a) = \mathbf{c} = \mathbf{y}(a)$.

Denote $\sigma(t) = \|\mathbf{x}(t) - \mathbf{y}(t)\|^2 = \sum_{k=1}^{n} [x_k(t) - y_k(t)]^2 \geq 0$. Then,

$$\begin{aligned}
\sigma'(t) &= \sum_{k=1}^{n} 2\left[x_k'(t) - y_k'(t)\right] \cdot [x_k(t) - y_k(t)] \\
&= \sum_{k=1}^{n} 2\left[X_k(\mathbf{x}(t), t) - X_k(\mathbf{y}(t), t)\right] \cdot [x_k(t) - y_k(t)] \\
&= 2\left[\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)\right] \cdot [\mathbf{x}(t) - \mathbf{y}(t)].
\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}
\sigma'(t) &\leq \left|\sigma'(t)\right| = 2\left|(\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)) \cdot (\mathbf{x} - \mathbf{y})\right| \\
&\leq 2\|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\| \cdot \|\mathbf{x} - \mathbf{y}\| \\
&\leq 2L\|\mathbf{x} - \mathbf{y}\|^2 = 2L\sigma(t).
\end{aligned}$$

The last inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma(t)e^{-2Lt}\right) = \left(\sigma'(t) - 2L\sigma(t)\right)e^{-2Lt} \leq 0.$$

Hence $\sigma(t)e^{-2Lt}$ is a decreasing function. Therefore for $t > a$, $\sigma(t)e^{-2Lt} \leq \sigma(a)e^{-2La} = 0$. Since $\sigma(t) \geq 0$, we have $\sigma(t) = 0$ for $t \geq a$, i.e., $\mathbf{x}(t) = \mathbf{y}(t)$ for $t \geq a$.

To argue for $t < a$, as above we again have

$$-\sigma'(t) \leq |\sigma'(t)| \leq 2L\sigma(t),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma(t)e^{+2Lt}\right) = \left(\sigma'(t) + 2L\sigma(t)\right)e^{+2Lt} \geq 0.$$

So $\sigma(t)e^{+2Lt}$ is an increasing function. Therefore for $t < a$, $\sigma(t)e^{+2Lt} \leq \sigma(a)e^{+2La} = 0$. Again we have $\sigma(t) = 0$ for $t \leq a$, i.e., $\mathbf{x}(t) = \mathbf{y}(t)$ for $t \leq a$. □

Example 1.2.1 *Find a region where the differential equation* $x' = x + 3x^{1/3}$ *has a unique solution, i.e., find* (x_0, t_0) *such that the solution* $x(t)$ *of the differential equation with* $x(t_0) = x_0$ *is unique in a neighborhood of* (x_0, t_0).

Solution We will show the following

(1) the differential equation has a unique solution with $x(t_0) = x_0$, $x_0 \neq 0$;

(2) there are more than one solution satisfying $x(t_0) = 0$.

(1) For any given (x_0, t_0), with $x_0 \neq 0$, we choose a small $\delta > 0$ such that $0 \notin [x_0 - \delta, x_0 + \delta]$. By Lemma 1.1.2, we know that the function $X(x,t) = x + 3x^{1/3}$ satisfies a Lipschitz condition in the region

$$\mathcal{R} = \left\{ (x,t) \;\middle|\; |x - x_0| \leq \delta, |t - t_0| \leq T \right\} = [x_0 - \delta, x_0 + \delta] \times [t_0 - T, t_0 + T],$$

where $T > 0$ is any fixed constant.[3] By Theorem 1.2.1, we conclude that the differential equation has a unique solution with $x(t_0) = x_0$, $x_0 \neq 0$.

(2) It is easy to see that $x(t) \equiv 0$ is one solution of the differential equation with $x(t_0) = 0$. We only need to show that there exists another solution which also satisfies $x(t_0) = 0$.

Consider the improper integral

$$\int_0^x \frac{\mathrm{d}u}{u + 3u^{1/3}}.$$

For any $c > 0$, we know that

$$0 < \int_0^c \frac{\mathrm{d}u}{u + 3u^{1/3}} < \int_0^c \frac{\mathrm{d}u}{3u^{1/3}} = \frac{1}{2}c^{2/3}.$$

Hence the improper integral converges for $c > 0$. This allows us to define an implicit function $x(t)$ by

$$\int_0^x \frac{\mathrm{d}u}{u + 3u^{1/3}} = t - t_0.$$

We can further assume that $x(t_0) = 0$, since the last equation becomes an identity when setting $t = t_0$. Obviously, this function $x(t) \not\equiv 0$, otherwise we will have $t \equiv t_0$, a contradiction. This function $x(t)$ certainly satisfies the differential equation, which can be seen easily by differentiating both sides of the last equation. □

[3] We can make the following argument. Let $(x,t), (y,t) \in \mathcal{R} = [x_0 - \delta, x_0 + \delta] \times [t_0 - T, t_0 + T]$. Then, by the mean value theorem, we have

$$|X(x,t) - X(y,t)| = \left| \frac{\partial X}{\partial x}(\xi, t) \right| \cdot |x - y| = \left| 1 - \xi^{-2/3} \right| \cdot |x - y|,$$

where ξ is between x and y. This implies $\xi \in [x_0 - \delta, x_0 + \delta]$. The function $\dfrac{\partial X}{\partial x}(x,t) = 1 + x^{-2/3}$ is continuous in x in the compact set $[x_0 - \delta, x_0 + \delta]$, since $0 \notin [x_0 - \delta, x_0 + \delta]$. This indicates that there is a number L (might depend on x_0 and δ) such that

$$\left| 1 - \xi^{-2/3} \right| \leq L.$$

Thus, the function $X(x,t)$ satisfies a Lipschitz condition in $[x_0 - \delta, x_0 + \delta] \times \mathbb{R}$.

1.3 CONTINUITY

Theorem 1.3.1 (**Continuity Theorem**) *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *be any two solutions of the differential equation (1.3) in* $T_1 \le t \le T_2$*, where* $\mathbf{X}(\mathbf{x}, t)$ *is continuous and satisfies the Lipschitz condition (1.2) in some region* $\mathcal{R}$ *that contains the region where* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are defined. Then*

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \le e^{L|t-a|} \|\mathbf{x}(a) - \mathbf{y}(a)\|,$$

for any $a, t \in [T_1, T_2]$.

Proof Let us first assume that $t \ge a$. Then, for the function $\sigma(t) = \|\mathbf{x}(t) - \mathbf{y}(t)\|^2$, as in the proof of Uniqueness Theorem 1.2.1, we have

$$\sigma'(t) \le 2L\sigma(t),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma(t)e^{-2Lt}\right) \le 0.$$

Integrating the last inequality from a to t gives

$$\sigma(t)e^{-2Lt} \le \sigma(a)e^{-2La},$$

which yields the desired inequality.

In the case $t \le a$, again as in the proof of Uniqueness Theorem 1.2.1, we obtain

$$-\sigma'(t) \le 2L\sigma(t),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma(t)e^{+2Lt}\right) \ge 0.$$

Now we integrate the last inequality from t to a to have

$$\sigma(t)e^{2Lt} \le \sigma(a)e^{2La}.$$

This implies the desired inequality in the case $t \le a$. □

Corollary 1.3.2 *Let* $\mathbf{x}(t)$ *be the solution of the differential equation (1.3) satisfying the initial condition* $\mathbf{x}(a, \mathbf{c}) = \mathbf{c}$*. Let the hypotheses of Continuity Theorem 1.3.3 be satisfied, and let the function* $\mathbf{x}(t, \mathbf{c})$ *be defined for* $\|\mathbf{c} - \mathbf{c}^0\| \le K$ *and* $|t - a| \le T$*. Then*

(1) $\mathbf{x}(t, \mathbf{c})$ *is a continuous function of both variables;*

(2) *if* $\mathbf{c} \to \mathbf{c}^0$*, then* $\mathbf{x}(t, \mathbf{c}) \to \mathbf{x}(t, \mathbf{c}^0)$ *uniformly for* $|t - a| \le T$.

Proof For (a), it is obvious that $\mathbf{x}(t, \mathbf{c})$ is continuous in t since $\mathbf{x}(t, \mathbf{c})$ is a solution of the differential equation (1.3), which is differentiable. To see that $\mathbf{x}(t, \mathbf{c})$ is continuous in $\mathbf{c}$, we take $\mathbf{c}^1$ and $\mathbf{c}^2$, with $\left\|\mathbf{c}^1 - \mathbf{c}^0\right\| \le K$ and $\left\|\mathbf{c}^2 - \mathbf{c}^0\right\| \le K$. Since $\mathbf{x}(t, \mathbf{c}^1)$ and $\mathbf{x}(t, \mathbf{c}^2)$ are solutions of (1.3), by Continuity Theorem 1.3.3, we get

$$\left\|\mathbf{x}(t, \mathbf{c}^1) - \mathbf{x}(t, \mathbf{c}^2)\right\| \le e^{L|t-a|} \left\|\mathbf{x}(a, \mathbf{c}^1) - \mathbf{x}(a, \mathbf{c}^2)\right\| = e^{L|t-a|} \left\|\mathbf{c}^1 - \mathbf{c}^2\right\|.$$

Let t be fixed. Hence, for any given $\epsilon > 0$, take $\delta = \dfrac{\epsilon}{e^{L|t-a|}}$, whenever $\left\|\mathbf{c}^1 - \mathbf{c}^2\right\| < \delta$, we have

$$\left\|\mathbf{x}(t, \mathbf{c}^1) - \mathbf{x}(t, \mathbf{c}^2)\right\| < e^{L|t-a|}\delta = \epsilon.$$

Therefore, $\mathbf{x}(t, \mathbf{c})$ is continuous in $\mathbf{c}$.

For (b), in above, we can take

$$\delta = \min_{t \in [a-T, a+T]} \left\{ \frac{1}{e^{L|t-a|}} \right\} \cdot \epsilon.$$

Since $\min_{t \in [a-T, a+T]} \left\{ \frac{1}{e^{L|t-a|}} \right\} > 0$ is a finite number, hence, $\delta > 0$ is independent of t. Therefore, $\mathbf{x}(t, \mathbf{c})$ is continuous in $\mathbf{c}$ uniformly for $|t-a| \leq T$. □

Theorem 1.3.3 (**Strong Continuity Theorem**) *Assume that*

(1) $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *satisfy the differential equations*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t) \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{Y}(\mathbf{y}, t),$$

respectively, on $T_1 \leq t \leq T_2$.

(2) *The functions* $\mathbf{X}$ *and* $\mathbf{Y}$ *be defined and continuous in a common domain* $\mathcal{R}$, *and satisfy*

$$\|\mathbf{X}(\mathbf{z}, t) - \mathbf{Y}(\mathbf{z}, t)\| \leq \epsilon,$$

for any $(\mathbf{z}, t) \in \mathcal{R}$, *with* $T_1 \leq t \leq T_2$.

(3) $\mathbf{X}(\mathbf{x}, t)$ *satisfies the Lipschitz condition (1.2) in* $\mathcal{R}$.[4]

Then

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq \|\mathbf{x}(a) - \mathbf{y}(a)\| \, e^{L|t-a|} + \frac{\epsilon}{L}\left(e^{L|t-a|} - 1\right),$$

for any $a, t \in [T_1, T_2]$.

Proof Assume that $a, t \in [T_1, T_2]$. Let us first consider the case $t \geq a$. Define $\sigma(t) = \|\mathbf{x}(t) - \mathbf{y}(t)\|^2 = \sum_{k=1}^{n} [x_k(t) - y_k(t)]^2 \geq 0$. We have

$$\begin{aligned}
\sigma'(t) &= 2\left[\mathbf{x}'(t) - \mathbf{y}'(t)\right] \cdot [\mathbf{x}(t) - \mathbf{y}(t)] \\
&= 2\left[\mathbf{X}(\mathbf{x}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)\right] \cdot [\mathbf{x}(t) - \mathbf{y}(t)] \\
&= 2\left[\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)\right] \cdot [\mathbf{x}(t) - \mathbf{y}(t)] + 2\left[\mathbf{X}(\mathbf{y}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)\right] \cdot [\mathbf{x}(t) - \mathbf{y}(t)].
\end{aligned}$$

The Schwarz inequality implies

$$\begin{aligned}
\sigma'(t) &\leq \left|\sigma'(t)\right| \\
&\leq 2\|\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)\| \cdot \|\mathbf{x}(t) - \mathbf{y}(t)\| + 2\|\mathbf{X}(\mathbf{y}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)\| \cdot \|\mathbf{x}(t) - \mathbf{y}(t)\| \\
&\leq 2L\|\mathbf{x}(t) - \mathbf{y}(t)\|^2 + 2\epsilon\|\mathbf{x}(t) - \mathbf{y}(t)\| \\
&= 2L\sigma(t) + 2\epsilon\sqrt{\sigma(t)}.
\end{aligned}$$

[4]The function $\mathbf{Y}(\mathbf{y}, t)$ may not satisfy a Lipschitz condition.

(1) Let us assume that $\sigma(a) > 0$. Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = 2Lu + 2\epsilon\sqrt{u}, & t \geq a, \\ u(a) = \sigma(a). \end{cases}$$

Since initially $u(a) > 0$, the right hand side of the differential equation is non-negative. So, u is an increasing function. Hence, for $t \geq a$, $u(t) \geq u(a) > 0$. Thus, we can introduce the substitution $v(t) = \sqrt{u(t)}$. This gives the equivalent differential equation

$$2v\frac{\mathrm{d}v}{\mathrm{d}t} = 2Lv^2 + 2\epsilon v.$$

Since $v(t) > 0$, we can divide both sides of the equation by $v(t)$ to obtain a linear differential equation for v. This leads to the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} - Lv = \epsilon, & t \geq a, \\ v(a) = \sqrt{u(a)}. \end{cases}$$

The solution of the initial value problem is

$$\sqrt{u(t)} = v(t) = \sqrt{u(a)}e^{L(t-a)} + \frac{\epsilon}{L}\left(e^{L(t-a)} - 1\right).$$

Since $u(a) = \sigma(a) > 0$, we have

$$\begin{aligned} [\sigma(t) - u(t)]' &\leq 2L[\sigma(t) - u(t)] + 2\epsilon[\sqrt{\sigma(t)} - \sqrt{u(t)}] \\ &= 2L[\sigma(t) - u(t)] + 2\epsilon \cdot \frac{\sigma(t) - u(t)}{\sqrt{\sigma(t)} + \sqrt{u(t)}} \\ &\leq 2L[\sigma(t) - u(t)] + 2\epsilon \cdot \frac{\sigma(t) - u(t)}{\sqrt{u(a)}}, \end{aligned}$$

which implies

$$\left[[\sigma(t) - u(t)] \cdot \exp\left\{-\left(2L + 2\epsilon/\sqrt{u(a)}\right)t\right\}\right]' \leq 0.$$

Integrating the last inequality from a to t, we have

$$[\sigma(t) - u(t)] \cdot \exp\left\{-\left(2L + 2\epsilon/\sqrt{u(a)}\right)t\right\} \leq [\sigma(a) - u(a)] \cdot \exp\left\{-\left(2L + 2\epsilon/\sqrt{u(a)}\right)a\right\} = 0.$$

Thus,

$$\sigma(t) \leq u(t),$$

which implies

$$\begin{aligned} \|\mathbf{x}(t) - \mathbf{y}(t)\| = \sqrt{\sigma(t)} &\leq \sqrt{u(t)} \\ &= \sqrt{u(a)}e^{L(t-a)} + \frac{\epsilon}{L}\left(e^{L(t-a)} - 1\right) \\ &= \sqrt{\sigma(a)}e^{L(t-a)} + \frac{\epsilon}{L}\left(e^{L(t-a)} - 1\right) \\ &= \|\mathbf{x}(a) - \mathbf{y}(a)\|\, e^{L(t-a)} + \frac{\epsilon}{L}\left(e^{L(t-a)} - 1\right). \end{aligned}$$

(2) For $\sigma(a) = 0$, we have to modify the discussion above. For each positive integer n, we consider the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = 2Lu + 2\epsilon\sqrt{u}, & t \geq a, \\ u(a) = 1/n. \end{cases}$$

The discussion above can be applied to this problem, and we have the solution

$$u_n(t) = \left[n^{-1/2}e^{L|t-a|} + \frac{\epsilon}{L}\left(e^{L|t-a|} - 1\right)\right]^2.$$

If we can show that $\sigma(t) \leq u_n(t)$ for $t \geq a$, then, after taking $n \to \infty$, we obtain the desired inequality in the case $\sigma(a) = 0$. The last inequality can be proved by contradiction. In fact, if

$$\sigma(t_1) > u_n(t_1),$$

for some $t_1 > a$, then there exists t_0 to be the largest t in the interval $a < t \leq t_1$ such that $\sigma(t_0) \leq u_n(t_0)$. Obviously, $\sigma(t_0) = u_n(t_0) > 0$ and $\sigma(t) > u_n(t)$ for $t_0 < t \leq t_1$. But this is impossible according to the discussion in the case $\sigma(a) = 0$.

We can use the inequality

$$-\sigma'(t) \leq |\sigma'(t)| \leq 2L\sigma(t) + 2\epsilon\sqrt{\sigma(t)}$$

and a similar argument to prove the following inequality

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq \|\mathbf{x}(a) - \mathbf{y}(a)\|\, e^{L(a-t)} + \frac{\epsilon}{L}\left(e^{L(a-t)} - 1\right).$$

□

1.4 EXISTENCE THEOREM

In this section, we study existence of the differential equation (1.3). The idea is to establish an equivalent integral equation for any given initial value problem. Then we show that the iteration of the integral operator converges to a solution.

Theorem 1.4.1 *Let $\mathbf{X}(\mathbf{x}, t)$ be a continuous function. Then a function $\mathbf{x}(t)$ is a solution of the initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t), \\ \mathbf{x}(a) = \mathbf{c} \end{cases} \tag{1.4}$$

if and only if it is a solution of the integral equation

$$\mathbf{x}(t) = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}(s), s)\mathrm{d}s. \tag{1.5}$$

Proof Let us assume that $\mathbf{x}(t)$ is a solution of the initial value problem (1.4). The Fundamental Theorem of Calculus implies that

$$x_k(t) = x_k(a) + \int_a^t x_k'(s)\mathrm{d}s.$$

Using (1.4), we have the integral equation (1.5).

Conversely, if $\mathbf{x}(t)$ is a solution of the integral equation (1.5), then $\mathbf{x}(a) = \mathbf{c}$ and, by the Fundamental Theorem of Calculus, we have

$$x_k'(t) = X_k(\mathbf{x}(t), t), \qquad k = 1, \ldots, n.$$

These imply that $\mathbf{x}(t)$ satisfies $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t)$. □

For a given $\mathbf{X}(\mathbf{x}, t)$, if it is defined for all $\mathbf{x}$ in $|t - a| \leq T$, and is continuous, then we can define an operator U by

$$U(\mathbf{x}) = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}(s), s)\mathrm{d}s. \tag{1.6}$$

For this operator, its domain is

$$\left\{\mathbf{x}(t) \middle| \ \mathbf{x}(t) \text{ is continuous in the interval } |t - a| \leq T\right\}$$

and its range is

$$\left\{\mathbf{y}(t) \middle| \ \mathbf{y}(t) \text{ is continuously differentiable in the interval } |t - a| \leq T \text{ and } \mathbf{y}(a) = \mathbf{c}\right\}.$$

Thus, a solution of the integral equation (1.5) is a fixed point of the operator U:

$$\mathbf{x} = U(\mathbf{x}).$$

Theorem 1.4.1 can be re-stated as the following:

Theorem 1.4.2 *Let $\mathbf{X}(\mathbf{x}, t)$ be a continuous function. Then a function $\mathbf{x}(t)$ is a solution of the initial value problem (1.4) if and only if the operator U defined in (1.6) has a fixed point in $C[a - T, a + T]$.*

Now we can use the operator U to generate a sequence of functions $\{\mathbf{x}^n\}$ from the given initial data by the successive iteration:

$$\mathbf{x}^0(t) \equiv \mathbf{c}, \qquad \mathbf{x}^n = U(\mathbf{x}^{n-1}) = U^n(\mathbf{x}^0), \qquad n = 1, 2, \ldots. \tag{1.7}$$

This is call a Picard iteration.[5]

Now we show that, under some conditions, a Picard iteration converges to a solution of the initial value problem (1.4).

Lemma 1.4.3 *Assume that $\mathbf{X}(\mathbf{x}, t)$ is continuous and satisfies the Lipschitz condition (1.2) on the interval $|t - a| \leq T$ for all $\mathbf{x}, \mathbf{y}$. Then the Picard iteration (1.7) converges uniformly for $|t - a| \leq T$.*

Proof Let $M = \sup_{|t-a| \leq T} |\mathbf{X}(\mathbf{c}, t)| < +\infty$. Without loss of generality we can assume that $a = 0$ and $t \geq a$. In other words, we prove the lemma on the interval $0 \leq t \leq T$. The proof for the general a and $t < a$ can be deduced from this case by the substitutions $t \to t + a$ and $t \to a - t$.

We first prove by induction that

$$\left\|\mathbf{x}^n(t) - \mathbf{x}^{n-1}(t)\right\| \leq \frac{(M/L)(Lt)^n}{n!}, \qquad n = 1, 2, \ldots. \tag{1.8}$$

[5] The Picard iteration is started from $\mathbf{x}^0 = \mathbf{c}$, given by the initial data.

In fact, for $n = 1$,

$$\begin{aligned}
\left\|\mathbf{x}^1(t) - \mathbf{x}^0(t)\right\| &= \left\|\int_0^t \mathbf{X}(\mathbf{x}^0(s), s)\ \mathrm{d}s\right\| \\
&\leq \int_0^t \left\|\mathbf{X}(\mathbf{x}^0(s), s)\right\| \mathrm{d}s \\
&\leq M \int_0^t \mathrm{d}s = Mt = \frac{(M/L)(Lt)^1}{1!}.
\end{aligned}$$

For $n = 2$,

$$\begin{aligned}
\left\|\mathbf{x}^2(t) - \mathbf{x}^1(t)\right\| &= \left\|\int_0^t \left[\mathbf{X}(\mathbf{x}^1(s), s) - \mathbf{X}(\mathbf{x}^0(s), s)\right] \mathrm{d}s\right\| \\
&\leq \int_0^t \left\|\mathbf{X}(\mathbf{x}^1(s), s) - \mathbf{X}(\mathbf{x}^0(s), s)\right\| \mathrm{d}s \\
&\leq L \int_0^t \left\|\mathbf{x}^1(s) - \mathbf{x}^0(s)\right\| \mathrm{d}s \\
&\leq L \int_0^t Ms\ \mathrm{d}s = \frac{LMt^2}{2} = \frac{(M/L)(Lt)^2}{2!}.
\end{aligned}$$

Assume that the desired inequality holds for $n = k$:

$$\left\|\mathbf{x}^k(t) - \mathbf{x}^{k-1}(t)\right\| \leq \frac{(M/L)(Lt)^k}{k!}.$$

Then,

$$\begin{aligned}
\left\|\mathbf{x}^{k+1}(t) - \mathbf{x}^k(t)\right\| &= \left\|\int_0^t \left[\mathbf{X}(\mathbf{x}^k(s), s) - \mathbf{X}(\mathbf{x}^{k-1}(s), s)\right] \mathrm{d}s\right\| \\
&\leq \int_0^t \left\|\mathbf{X}(\mathbf{x}^k(s), s) - \mathbf{X}(\mathbf{x}^{k-1}(s), s)\right\| \mathrm{d}s \\
&\leq L \int_0^t \left\|\mathbf{x}^k(s) - \mathbf{x}^{k-1}(s)\right\| \mathrm{d}s \\
&\leq L \int_0^t \frac{(M/L)(Ls)^k}{k!}\ \mathrm{d}s = \frac{(M/L)(Lt)^{k+1}}{(k+1)!}.
\end{aligned}$$

Hence, the estimates (1.8) hold.

Next, we show that the sequence $\{\mathbf{x}^n(t)\}$ converges uniformly for $0 \leq t \leq T$. Indeed, for $0 \leq t \leq T$, since

$$\frac{(M/L)(Lt)^k}{k!} \leq \frac{(M/L)(LT)^k}{k!},$$

and the positive series

$$\sum_{k=1}^{\infty} \frac{(M/L)(LT)^k}{k!}$$

is convergent to $(M/L)\left(e^{LT} - 1\right)$, by the Comparison Test, the series

$$\mathbf{x}^0(t) + \sum_{k=1}^{\infty} \left[\mathbf{x}^k(t) - \mathbf{x}^{k-1}(t)\right]$$

is also convergent. Actually, the convergence is uniform for $0 \le t \le T$, since $\dfrac{(M/L)(LT)^k}{k!}$ is independent of t. The n-th partial sum,

$$\mathbf{x}^0(t) + \sum_{k=1}^{n} \left[\mathbf{x}^k(t) - \mathbf{x}^{k-1}(t)\right] = \mathbf{x}^n(t).$$

Therefore, the sequence of functions $\{\mathbf{x}^n(t)\}$ converges uniformly. □

Theorem 1.4.4 (**Existence Theorem**) *Assume that $\mathbf{X}(\mathbf{x}, t)$ is continuous and satisfies the Lipschitz condition (1.2) on the interval $|t - a| \le T$ for all $\mathbf{x}, \mathbf{y}$. Then the initial value problem (1.4) has a unique solution on the interval $|t - a| \le T$.*

Proof The uniqueness is a direct consequence of Uniqueness Theorem 1.2.1. We only need to prove the existence.

By Lemma 1.4.3, the sequence $\{\mathbf{x}^n(t)\}$ defined by the Picard iteration with $\mathbf{x}^0(t) \equiv \mathbf{c}$ is uniformly convergent. Denote $\mathbf{x}^\infty(t)$ the limit function. We show that $\mathbf{x}^\infty(t)$ is a solution of the integral equation (1.5). By the definition,

$$\mathbf{x}^{n+1}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^n(s), s)\ \mathrm{d}s.$$

The left hand side is uniformly convergent to $\mathbf{x}^\infty(t)$. By the Lipschitz condition,

$$\|\mathbf{X}(\mathbf{x}^m(s), s) - \mathbf{X}(\mathbf{x}^n(s), s)\| \le L\, \|\mathbf{x}^m(s) - \mathbf{x}^n(s)\|,$$

and so the integral on the right hand side is also uniformly convergent. Since $\mathbf{X}(\mathbf{x}, s)$ is continuous, we know that

$$\mathbf{X}(\mathbf{x}^n(s), s) \to \mathbf{X}(\mathbf{x}^\infty(s), s).$$

Hence, we obtain

$$\mathbf{x}^\infty(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^\infty(s), s)\ \mathrm{d}s.$$

Finally, by Theorem 1.4.1, we conclude that the function $\mathbf{x}^\infty(t)$ is a solution of the initial value problem (1.4). □

Example 1.4.1 *Solve the initial value problem*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \qquad x(0) = 1.$$

by the Picard iteration.

Solution For this initial value problem, the integral operator U is defined as

$$U(x(t)) = 1 + \int_0^t x(s)\ \mathrm{d}s$$

Thus, the Picard iteration gives

$$
\begin{aligned}
x_0(t) &= 1, \\
x_1(t) &= U(x_0(t)) = 1 + \int_0^t 1 \, \mathrm{d}s = 1 + t, \\
x_2(t) &= U(x_1(t)) = 1 + \int_0^t (1+s) \, \mathrm{d}s \\
&= 1 + t + \frac{t^2}{2}, \\
&\vdots \\
x_n(t) &= 1 + t + \frac{t^2}{2} + \ldots + \frac{t^n}{n!}, \\
x_{n+1}(t) &= U(x_n(t)) \\
&= 1 + \int_0^t \left(1 + s + \frac{s^2}{2} + \ldots + \frac{s^n}{n!}\right) \, \mathrm{d}s \\
&= 1 + t + \frac{t^2}{2} + \ldots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!}.
\end{aligned}
$$

We see that the sequence $\{x_n(t)\}$ converges uniformly to the function e^t. Hence we get the solution $x(t) = e^t$ by the Picard iteration. □

Example 1.4.2 *Verify the Taylor series for* $\sin t$ *and* $\cos t$ *by applying the Picard iteration to the first order system corresponding to the second order initial value problem*

$$x'' = -x, \quad x(0) = 0, \quad x'(0) = 1.$$

Solution The associated first order system is

$$
\begin{cases}
\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \\
\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1,
\end{cases}
$$

with the initial condition

$$\mathbf{x}(0) = (x_1(0), x_2(0)) = (0, 1).$$

The corresponding $\mathbf{X}(\mathbf{x}, t) = (x_2, -x_1)$, and the initial value $\mathbf{c} = (0, 1)$. The Picard iteration yields

$$
\begin{aligned}
\mathbf{x}^0(t) &= (0, 1), \\
\mathbf{x}^1(t) &= (0, 1) + \int_0^t (1, 0) \, \mathrm{d}s \\
&= (0, 1) + (t, 0) = (t, 1), \\
\mathbf{x}^2(t) &= (0, 1) + \int_0^t (1, -s) \, \mathrm{d}s \\
&= (0, 1) + \left(t, -\frac{t^2}{2}\right) = \left(t, 1 - \frac{t^2}{2}\right),
\end{aligned}
$$

$$\begin{aligned}
\mathbf{x}^3(t) &= (0,1) + \int_0^t \left(1 - \frac{s^2}{2}, -s\right) \mathrm{d}s \\
&= (0,1) + \left(t - \frac{t^3}{3!}, -\frac{t^2}{2}\right) \\
&= \left(t - \frac{t^3}{3!}, 1 - \frac{t^2}{2}\right), \\
\mathbf{x}^4(t) &= (0,1) + \int_0^t \left(1 - \frac{s^2}{2}, -s + \frac{s^3}{3!}\right) \mathrm{d}s \\
&= (0,1) + \left(t - \frac{t^3}{3!}, -\frac{t^2}{2} + \frac{t^4}{4!}\right) \\
&= \left(t - \frac{t^3}{3!}, 1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right).
\end{aligned}$$

We claim that for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\mathbf{x}^{2n+1}(t) &= \left(t + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!}, 1 + \cdots + (-1)^n \frac{t^{2n}}{(2n)!}\right), \\
\mathbf{x}^{2n+2}(t) &= \left(t + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!}, 1 + \cdots + (-1)^{n+1} \frac{t^{2n+2}}{(2n+2)!}\right).
\end{aligned}$$

We prove this Claim by mathematical induction. When $n = 0$, they hold due to the calculations above. Suppose they hold when $n = k$. Then we have

$$\begin{aligned}
\mathbf{x}^{2k+3}(t) &= (0,1) + \int_0^t \left(1 + \cdots + (-1)^{k+1} \frac{s^{2k+2}}{(2n+2)!}, -s - \cdots - (-1)^k \frac{s^{2k+1}}{(2k+1)!}\right) \mathrm{d}s \\
&= \left(t + \cdots + (-1)^{k+1} \frac{t^{2k+3}}{(2k+3)!}, 1 + \cdots + (-1)^{k+1} \frac{t^{2k+2}}{(2k+2)!}\right), \\
\mathbf{x}^{2k+4}(t) &= (0,1) + \int_0^t \left(1 + \cdots + (-1)^{k+1} \frac{s^{2k+2}}{(2k+2)!}, -s - \cdots - (-1)^{k+1} \frac{s^{2k+3}}{(2k+3)!}\right) \mathrm{d}s \\
&= \left(t + \cdots + (-1)^{k+1} \frac{t^{2k+3}}{(2k+3)!}, 1 + \cdots + (-1)^{k+2} \frac{t^{2k+4}}{(2k+4)!}\right).
\end{aligned}$$

That is, the claim is also true for $n = k + 1$.

It can be easily shown that the associated initial value problem has a unique solution

$$\mathbf{x}(t) = (\sin t, \cos t).$$

The uniqueness can be verified by checking that the conditions of Uniqueness Theorem 1.2.1 hold for the associated problem. Since $\{\mathbf{x}^n(t)\}$ converges to the unique solution by Existence Theorem 1.4.4, we have

$$\begin{aligned}
\sin t &= t - \frac{t^3}{3!} + \cdots + (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} + \cdots, \\
\cos t &= 1 - \frac{t^2}{2!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + \cdots.
\end{aligned}$$

□

Example 1.4.3 *Let $f(x,t)$ be a function satisfying the Lipschitz condition $|f(x,t) - f(y,t)| \leq L|x-y|$ for all $0 \leq t \leq T$ and all x and y. Suppose $f(x,t)$ is continuous and bounded. Let $M = \sup\limits_{\substack{-\infty<x<\infty \\ 0\leq t\leq T}} |f(x,t)|$. Let $x(t)$ be a solution of $x' = f(x,t)$ with initial value $x(0) = c$ and $x_k(t)$ be the k-th term in the Picard's approximation method. Prove*

$$|x(t) - x_k(t)| \leq \frac{ML^k}{(k+1)!} t^{k+1},$$

for $t \geq 0$.

Solution: Use mathematical induction. When $k = 0$, we have

$$|x(t) - x_0(t)| = |x(t) - x(0)| = |x'(\xi)|$$

for some $0 < \xi < t$ by the Mean Value Theorem. Since $|x'(\xi)| = |f(\xi, t)| \leq M$, the desired inequality holds for $k = 0$.

Suppose the inequality holds for $k = n$. Since

$$\begin{aligned} |x(t) - x_{n+1}(t)| &= \left| \int_0^t f(x(s), s) ds - \int_0^t f(x_n(s), s) \mathrm{d}s \right| \\ &\leq \int_0^t |f(x(s), s) - f(x_n(s), s)| \, \mathrm{d}s \\ &\leq \int_0^t L|x(s) - x_n(s)| \mathrm{d}s \\ &\leq \int_0^t L \frac{ML^n}{(n+1)!} s^{n+1} \mathrm{d}s \\ &= ML^{n+1} \frac{t^{n+2}}{(n+2)!}, \end{aligned}$$

we know that the inequality holds when $k = n+1$. By mathematical induction, the inequality holds for all $k \geq 0$. □

Exercise 1.4

1. For the initial value problem $x' = tx$, $x(0) = 1$, obtain the n-th approximation of the Picard iteration. Use mathematical induction to justify the formula.

2. For the initial value problem $x' = t + x$, $x(0) = 0$, obtain $x_0(t)$, $x_1(t)$, $x_2(t)$ and the n-th term of the sequence of the Picard approximation. Use mathematical induction to justify the formula.

3. Assume that $\mathbf{X}(\mathbf{x}, t)$ is continuous and satisfies the Lipschitz condition (1.2) on the interval $|t-a| \leq T$ for all $\mathbf{x}, \mathbf{y}$. For any function $\mathbf{f}(t)$, continuous on $|t-a| \leq T$, define the sequence $\{\mathbf{x}^n(t)\}$ by

$$\mathbf{x}^0(t) = \mathbf{f}(t), \qquad \mathbf{x}^n(t) = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}^{n-1}(s), s) \mathrm{d}s, \qquad n = 1, 2, \ldots.$$

Show that $\{\mathbf{x}^n(t)\}$ is uniformaly convergent to the unique solution of the initial value problem (1.4).

4. Show that the initial value problem

$$\begin{cases} y'' + y^2 - 1 = 0, \\ y(0) = y_0, \\ y'(0) = y_1 \end{cases}$$

is equivalent to the integral equation

$$y(t) = y_0 + y_1 t - \int_0^t (t-s)\left[y^2(s) - 1\right] ds.$$

5. Solve the integral equation

$$y(t) = \cos t + \int_0^t \sin(\tau - t) \cdot y(\tau) d\tau.$$

6. (1) Show that the initial value problem

$$\begin{cases} y'' + (1+t^2)y = 0, & t > 0, \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

is equivalent to the integral equation

$$y(t) = \cos t + \int_0^t \sin(\tau - t) \cdot \tau^2 y(\tau) d\tau.$$

(2) Show that the successive approximations $\{\phi_n(t)\}$ defined by

$$\begin{cases} \phi_0(t) = 0, \\ \phi_n(t) = \cos t + \int_0^t \sin(\tau - t) \cdot \tau^2 \phi_{n-1}(\tau) d\tau, & n = 1, 2, \ldots \end{cases}$$

converges uniformly to the solution in (1) for $0 \le t < T$. Here T is any fixed constant.

1.5 LOCAL EXISTENCE THEOREM AND THE PEANO THEOREM

1.5.1 Local Existence Theorem

In the Existence Theorem 1.4.4, the function $\mathbf{X}(\mathbf{x}, t)$ satisfies a Lipschitz condition (1.2) for all $\mathbf{x}$. This condition is quite strong and many functions may fail this condition.

Example 1.5.1 *Show that the conclusion of Theorem 1.4.4 fails for the initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = e^x, \\ x(0) = c. \end{cases}$$

Solution The solution of the initial value problem is given implicitly by

$$e^{-c} - e^{-x} = t.$$

It is obvious that the function is defined only in $-\infty < t < e^{-c}$. Hence, there is no $\epsilon > 0$ such that the differential equation has a solution defined on all of $|t| < \epsilon$ for every initial value, since we can always take a sufficient large $c >$ such that $e^{-c} < \epsilon$. Thus, the conclusion of Theorem 1.4.4 fails for this initial value problem.

The cause of this failure is that the function $X(x) = e^x$ does not satisfy a Lipschitz condition for all x. In fact,

$$\frac{|X(x,t) - X(0,t)|}{|x - 0|} = \frac{e^x - 1}{x}$$

is unbounded for large values of x. □

However, if the function $\mathbf{X}(\mathbf{x}, t)$ satisfies a Lipschitz condition in a bounded domain, then a solution exists in a limited region.

Theorem 1.5.1 (**Local Existence Theorem**) *Assume that* $\mathbf{X}(\mathbf{x}, t)$ *is continuous and satisfies the Lipschitz condition (1.2) in the closed domain* $\|\mathbf{x} - \mathbf{c}\| \leq K$, $|t - a| \leq T$. *Then the initial value problem (1.4) has a unique solution in the interval* $|t - a| \leq \min\{T, K/M\}$, *where*

$$M = \sup_{\substack{\|\mathbf{x}-\mathbf{c}\| \leq K \\ |t-a| \leq T}} \|\mathbf{X}(\mathbf{x}, t)\| .$$

Proof The existence can be proved as in the proof of Theorem 1.4.4, except that we have to modify the estimates for the sequence $\{\mathbf{x}^n(t)\}$ by showing that if $\mathbf{x}(t)$ is defined and continuous on $|t-a| \leq \min\{T, K/M\}$ satisfying

(a) $\mathbf{x}(t)$ is defined and continuous on $|t - a| \leq \min\{T, K/M\}$;

(b) $\mathbf{x}(a) = \mathbf{c}$;

(c) $\|\mathbf{x}(t) - \mathbf{c}\| \leq K$ on $|t - a| \leq \min\{T, K/M\}$,

then one iteration $\mathbf{y} = U(\mathbf{x})$ still satisfies these three conditions. Indeed, (a) and (b) are obvious. To show (c), we have the following

$$\begin{aligned} \|\mathbf{y}(t) - \mathbf{c}\| &= \left\| \int_a^t \mathbf{X}(\mathbf{x}(s), s)\ \mathrm{d}s \right\| \\ &\leq \left| \int_a^t \|\mathbf{X}(\mathbf{x}(s), s)\|\ \mathrm{d}s \right| \\ &\leq M \cdot |t - a| \leq M \cdot \frac{K}{M} = K. \end{aligned}$$

□

1.5.2 The Peasno Theorem

The Lipschitz condition plays an important role in the proof of the Local Existence Theorem. However, if we drop it, we are still able to prove the existence, by a more sophisticated argument.

Theorem 1.5.2 (**Peano Existence Theorem**) *Assume that* $\mathbf{X}(\mathbf{x}, t)$ *is continuous in the closed domain* $\|\mathbf{x} - \mathbf{c}\| \leq K$, $|t - a| \leq T$. *Then the initial value problem (1.4) has at least one solution in the interval* $|t - a| \leq \min\{T, K/M\}$, *where*

$$M = \sup_{\substack{\|\mathbf{x}-\mathbf{c}\| \leq K \\ |t-a| \leq T}} \|\mathbf{X}(\mathbf{x}, t)\| .$$

To prove this theorem, we need the following definition and the famous Arzelá-Ascoli Theorem.

Definition 1.5.3 *A family of functions $\mathcal{F}$ is said to be equicontinuous on $[a,b]$ if for any given $\epsilon > 0$, there exists a number $\delta > 0$ such that*

$$\|\mathbf{x}(t) - \mathbf{x}(s)\| < \epsilon$$

whenever $|t-s| < \delta$ for every function $\mathbf{x} \in \mathcal{F}$ and $t, s \in [a,b]$.

Arzelá-Ascoli Theorem *Assume that the sequence $\{\mathbf{x}^n(t)\}$ is bounded and equicontinuous on $[a,b]$. Then there exists a subsequence $\{\mathbf{x}^{n_i}(t)\}$ that is uniformly convergent on $[a,b]$.*

Now we are ready to prove Peano Existence Theorem 1.5.2.

Proof Denote $T_1 = \min\{T, K/M\}$. As argued at the beginning of the proof of Lemma 1.4.3, we only need to prove the theorem on $0 \le t \le T_1$, with $a = 0$.

We first construct a sequence of bounded equicontinuous functions $\{\mathbf{x}^n(t)\}$ on $[0, T_1]$. For each n, define

$$\mathbf{x}^n(t) = \begin{cases} \mathbf{c}, & \text{for } 0 \le t \le T_1/n, \\ \mathbf{c} + \displaystyle\int_0^{t - T_1/n} \mathbf{X}\left(\mathbf{x}^n(s), s\right) \mathrm{d}s, & \text{for } T_1/n < t \le T_1. \end{cases}$$

The above formula defines the value of $\mathbf{x}^n(t)$ recursively in terms of the previous values of $\mathbf{x}^n(t)$.

We can use mathematical induction to show that

$$\|\mathbf{x}^n(t) - \mathbf{c}\| \le K$$

on $[0, T_1]$. Indeed, on $[0, T_1/n]$, it is trivial since $\mathbf{x}^n(t) = \mathbf{c}$. If we assume that the inequality holds on $[0, k \cdot T_1/n]$ $(0 \le k < n)$, then on $[k \cdot T_1/n, (k+1) \cdot T_1/n]$,

$$\begin{aligned} \|\mathbf{x}^n(t) - \mathbf{c}\| &= \left\| \int_0^{t - T_1/n} \mathbf{X}\left(\mathbf{x}^n(s), s\right) \mathrm{d}s \right\| \\ &\le M \cdot |t - T_1/n| \le M \cdot T_1 \le K. \end{aligned}$$

Hence, the sequence $\{\mathbf{x}^n(t)\}$ is uniformly bounded on $[0, T_1]$:

$$\|\mathbf{x}^n(t)\| \le \|\mathbf{c}\| + K.$$

The equicontinuity of the sequence $\{\mathbf{x}^n(t)\}$ on $[0, T_1]$ can be proven by the following estimates: for any $t_1, t_2 \in [0, T_1]$,

$$\begin{aligned} \|\mathbf{x}^n(t_1) - \mathbf{x}^n(t_2)\| &= \begin{cases} 0, & \text{if } t_1, t_2 \in [0, T_1/n], \\ \left\| \displaystyle\int_0^{t_2 - T1/n} \mathbf{X}(\mathbf{x}^n(s), s) \mathrm{d}s \right\|, & \text{if } t_1 \in [0, T_1/n] \text{ and } t_2 \in (T_1/n, T_1], \\ \left\| \displaystyle\int_0^{t_1 - T1/n} \mathbf{X}(\mathbf{x}^n(s), s) \mathrm{d}s \right\|, & \text{if } t_2 \in [0, T_1/n] \text{ and } t_1 \in (T_1/n, T_1], \\ \left\| \displaystyle\int_{t_1 - T1/n}^{t_2 - T1/n} \mathbf{X}(\mathbf{x}^n(s), s) \mathrm{d}s \right\|, & \text{if } t_1, t_2 \in (T_1/n, T_1] \end{cases} \\ &\le M|t - s|. \end{aligned}$$

By Arzelá-Ascoli Theorem, we know that there exists a uniformly convergent subsequence $\{\mathbf{x}^{n_i}(t)\}$ that converges to a continuous function $\mathbf{x}^{\infty}(t)$ on $[0, T_1]$ as $n_i \to \infty$. We can show that the function $\mathbf{x}^{\infty}(t)$ is actually a solution of the initial value problem (1.4). Indeed, for any fixed $t \in (0, T_1]$, we take n_i sufficiently large such that $T_1/n_i < t$. Thus, by the definition of $\{\mathbf{x}^n(t)\}$, we have

$$\mathbf{x}^{n_i}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\,\mathrm{d}s - \int_{t-T_1/n_i}^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\,\mathrm{d}s.$$

As $n_i \to \infty$, since $\mathbf{X}(\mathbf{x}, t)$ is uniformly continuous, we have $\int_0^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\,\mathrm{d}s \to \int_0^t \mathbf{X}(\mathbf{x}^{\infty}(s), s)\,\mathrm{d}s$; the second integral of the last equation tends to zero, since

$$\left|\int_{t-T_1/n_i}^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\,\mathrm{d}s\right| \le \int_{t-T_1/n_i}^t M\mathrm{d}s = M \cdot \frac{T_1}{n_i} \to 0.$$

Hence, we know that the function $\mathbf{x}^{\infty}(t)$ satisfies the integral equation

$$\mathbf{x}^{\infty}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^{\infty}(s), s)\,\mathrm{d}s.$$

□

Without the Lipschitz condition, it is known that solution of initial value problem is not necessarily unique. For instance, it can be earily see that the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{3}{2}x^{1/3}, & t \ge 0, \\ x(0) = 0, \end{cases}$$

is not unique, with the following two solutions

$$x_1(t) = 0, \qquad x_2(t) = t^{3/2}.$$

Exercise 1.5

1. Determine the existence region of the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = t^2 + x^2, \\ x(0) = 0, \end{cases}$$

where the equation is defined in the region $\mathcal{R} = \{(x,t) | |x| \le 1, |t| \le 1\}$. Find the first four approximations $x_0(t), x_1(t), x_2(t), x_3(t)$ of the Picard iteration.

2. Determine the existence region of the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = t^2 - x^2, \\ x(0) = 0, \end{cases}$$

where the equation is defined in the region $\mathcal{R} = \{(x,t) | |x| \le 1, |t| \le 1\}$. Find the first four approximations $x_0(t), x_1(t), x_2(t), x_3(t)$ of the Picard iteration.

3. Assume that the functions $F(x_1, x_2, \ldots, x_n, t)$ is continuous and satisfies a Lipschitz condition in $|t-a| \le T$, $\|\mathbf{x}-\mathbf{c}\| \le K$. Then the initial value problem

$$\begin{cases} u^{(n)} = F(u, u', u', \ldots, u^{(n-1)}, t), \\ u^{(i)}(a) = c_i, \qquad 0 \le i \le n-1, \end{cases}$$

has a unique solution on $|t-a| \le \min\{T, K/M\}$, where

$$M = \sup_{\substack{\|\mathbf{x}-\mathbf{c}\| \le K \\ |t-a| \le T}} \left(x_2^2 + \cdots + x_n^2 + F^2(x_1, x_2, \ldots, x_n, t)\right)^{1/2}.$$

1.6 LINEAR SYSTEMS

In this section, we apply the theorems obtained in the previous sections to linear differential systems of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + b_i(t), \qquad 1 \le i \le n, \tag{1.9}$$

Or, in a matrix form, we write

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t), \tag{1.10}$$

where $\mathbf{A}(t) = (a_{ij}(t))$ and $\mathbf{b}(t) = (b_1(t), \ldots, b_n(t))$.

Theorem 1.6.1 *Assume that the functions $a_{ij}(t)$ and $b_i(t)$ are continuous for $|t-a| \le T$, $1 \le i, j \le n$. The the initial value problem (1.9) with $\mathbf{x}(a) = \mathbf{c} = (c_1, \ldots, c_n)$ has a unique solution on $|t-a| \le T$.*

Proof The initial value problem can be re-written in terms of the vector form (1.4), with

$$\mathbf{X}(\mathbf{x}(t), t) = \left(\sum_{j=1}^{n} a_{1j}(t)x_j(t) + b_1(t), \ldots, \sum_{j=1}^{n} a_{nj}(t)x_j(t) + b_n(t)\right),$$

and

$$\mathbf{x}(a) = (c_1, \ldots, c_n).$$

The function $\mathbf{X}(\mathbf{x}, t)$ is continuous and satisfies the Lipschitz condition (1.2) on the interval $|t-a| \le T$ for all $\mathbf{x}, \mathbf{y}$. Indeed, by the Schwarz inequality, we have

$$\begin{aligned} & \|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)\|^2 \\ = & \left(\sum_{j=1}^{n} a_{1j}(t)(x_j - y_j)\right)^2 + \cdots + \left(\sum_{j=1}^{n} a_{nj}(t)(x_j - y_j)\right)^2 \\ \le & \left(\sum_{j=1}^{n} |a_{1j}(t)|^2\right) \cdot \left(\sum_{j=1}^{n} |x_j - y_j|^2\right) + \cdots + \left(\sum_{j=1}^{n} |a_{nj}(t)|^2\right) \cdot \left(\sum_{j=1}^{n} |x_j - y_j|^2\right) \\ = & \left(\sum_{i,j} |a_{ij}(t)|^2\right) \cdot \|\mathbf{x}-\mathbf{y}\|^2 \\ \le & \left(\sum_{i,j} \sup_{|t-a| \le T} |a_{ij}(t)|^2\right) \cdot \|\mathbf{x}-\mathbf{y}\|^2. \end{aligned}$$

Hence the function $\mathbf{X}(\mathbf{x}, t)$ satisfies the conditions in Existence Theorem 1.4.4. This gives the existence of the solution. The uniqueness follows from Uniqueness Theorem 1.2.1. □

If $b_i(t) \equiv 0$, $1 \le i \le n$, the linear differential system (1.9) is called *homogeneous*, for which we can construct its general solution.

Assume that $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are n solutions of a homogeneous linear differential system. Obiously, the matrix

$$\mathbf{\Psi}(t) = \big(\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)\big) = \begin{pmatrix} x_1^1(t) & \cdots & x_1^n(t) \\ \vdots & & \vdots \\ x_n^1(t) & \cdots & x_n^n(t) \end{pmatrix}, \tag{1.11}$$

satisfies

$$\frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t).$$

If these n solutions are linearly independent at every point t, $\mathbf{\Psi}(t)$ is called a *fundamental matrix* of the homogeneous linear differential system.

Theorem 1.6.2 *Assume that the functions $a_{ij}(t)$ are continuous for $|t-a| \le T$, $1 \le i, j \le n$. If $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are k solutions of the homogeneous linear differential system, then the constant vectors $\mathbf{x}^1(t_0), \ldots, \mathbf{x}^k(t_0)$ are linearly independent for some t_0 if and only if they are linearly independent for every t on $|t-a| \le T$.*

Proof Assume that $\mathbf{x}^1(t_0), \ldots, \mathbf{x}^k(t_0)$ are linearly independent for some t_0. Let $\tilde{t}$ be another point on $|t-a| \le T$ other than t_0. We need to show that if

$$\alpha_1 \mathbf{x}^1(\tilde{t}) + \cdots + \alpha_n \mathbf{x}^k(\tilde{t}) = \mathbf{0},$$

then, $\alpha_1 = \ldots = \alpha_k = 0$. In fact, since $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are solutions, we have two solutions of the homogeneous system, $\mathbf{0}$ and

$$\alpha_1 \mathbf{x}^1 + \cdots + \alpha_n \mathbf{x}^k,$$

both vanishing at $\tilde{t}$. By the uniqueness of Theorem 1.6.1, we conclude that

$$\alpha_1 \mathbf{x}^1(t) + \cdots + \alpha_n \mathbf{x}^k(t) = \mathbf{0}$$

for every t on $|t-a| \le T$. In particular, we have

$$\alpha_1 \mathbf{x}^1(t_0) + \cdots + \alpha_n \mathbf{x}^k(t_0) = \mathbf{0}.$$

The linear independence of $\mathbf{x}^1(t_0), \ldots, \mathbf{x}^k(t_0)$ implies that all $\alpha_i = 0$, $1 \le i \le k$.

The other direction is obvious. □

Theorem 1.6.3 *Assume that the functions $a_{ij}(t)$ are continuous for $|t-a| \le T$, $1 \le i, j \le n$. Let $\mathbf{x}^i(t)$, $1 \le i \le n$, be n solutions of the homogeneous linear differential system (1.9) and assume that they are linearly independent at some t_0. Define $\mathbf{\Psi}(t) = \big(\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)\big)$. Then the solution of the homogeneous linear differential system satisfying the initial condition $\mathbf{x}(a) = \mathbf{c} = (c_1, \ldots, c_n)$ is given by*

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(a)\mathbf{c}.$$

In particular, if $\mathbf{x}^i(t)$ *be the solution of the homogeneous linear differential system (1.9) that satisfies the initial condition* $x_k^i(a) = 0$, $i \neq 0$, $x_i^i(a) = 1$. *The the solution satisfying the initial condition* $\mathbf{x}(a) = \mathbf{c} = (c_1, \ldots, c_n)$ *is given by*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + \cdots + c_n\mathbf{x}^n(t).$$

Proof Since $\mathbf{x}^1(t_0), \ldots, \mathbf{x}^n(t_0)$ are linearly independent, by Theorem 1.6.2, we know that the matrix $\mathbf{\Psi}(t) = \big(\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)\big)$ is non-singular for every t on $|t-a| \le T$ and hence, $\mathbf{\Psi}(t)$ is a fundamental matrix. Let $\mathbf{x}(t)$ be the unique solution of the homogeneous system satisfying the initial condition $\mathbf{x}(a) = \mathbf{c}$. Its existence is guaranteed by Theorem 1.6.1. Consider the function

$$\mathbf{x}(t) - \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(a)\mathbf{c}.$$

Since $\mathbf{\Psi}(t)$ is a fundamental matrix, we know this function is again a solution of the homogeneous system. Now we have two solutions of the homogeneous system, $\mathbf{0}$ and $\mathbf{x}(t) - \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(a)\mathbf{c}$, both vanishing at $t = a$. By the uniqueness of Theorem 1.6.1, we have

$$\mathbf{x}(t) - \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(a)\mathbf{c} = \mathbf{0},$$

That is, $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(a)\mathbf{c}$.

In particular, if $\mathbf{x}^i(t)$ satisfies the initial condition $x_k^i(a) = 0$, $i \neq 0$, $x_i^i(a) = 1$, then $\mathbf{\Psi}(a)$ is the identity matrix. Therefore

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} = c_1\mathbf{x}^1(t) + \cdots + c_n\mathbf{x}^n(t).$$

□

Theorem 1.6.4 *Let* $\mathbf{A}(t) = (a_{ij}(t))$ *and* $\mathbf{b}(t)$ *be continuous on* $|t-a| \le T$. *The solution of the non-homogeneous linear system (1.10), with initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$, *is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{b}(s)\mathrm{d}s.$$

where $\mathbf{\Phi}(t)$ *is any fundamental matrix of the corresponding homogeneous system. In particular, if* $\mathbf{A}$ *is a constant matrix, then*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)\mathbf{b}(s)\mathrm{d}s.$$

Furthermore, if $\mathbf{\Psi}(t)$ *is the fundamental matrix satisfying* $\mathbf{\Psi}(0) = I$, *then*

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Psi}(t-s)\mathbf{b}(s)\mathrm{d}s.$$

Proof For any given fundamental matrix $\mathbf{\Phi}(t)$ of the homogeneous system, we postulate the solution $\mathbf{x}(t)$ of the nonhomogeneous system satisfying the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ to be in the form

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\left\{\mathbf{x}_0 + \phi(t)\right\}.$$

Then, $\phi(t)$ satisfies the initial condition $\phi(t_0) = \mathbf{0}$. To find the equation satisfied by $\phi(t)$, we substitute the expression into the nonhomogeneous system to have

$$\mathbf{\Phi}'(t)\mathbf{\Phi}^{-1}(t_0)\left\{\mathbf{x}_0 + \phi(t)\right\} + \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\phi'(t) = \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\left\{\mathbf{x}_0 + \phi(t)\right\} + \mathbf{b}(t).$$

Since $\mathbf{\Phi}'(t) = \mathbf{A}(t)\mathbf{\Phi}(t)$, the last equation gives

$$\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\phi'(t) = \mathbf{b}(t),$$

whose solution satisfying $\phi(t_0) = \mathbf{0}$ is

$$\phi(t) = \mathbf{\Phi}(t_0)\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{b}(s)\mathrm{d}s.$$

Thus

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{b}(s)\mathrm{d}s.$$

If $\mathbf{A}$ is a constant matrix, to show

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)\mathbf{b}(s)\mathrm{d}s,$$

it is sufficient to show that

$$\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0),$$

for any t, s and t_0. Indeed, let $U(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$ and $V(t) = \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)$, with s and t_0 being two parameters. It is obvious that $U(s) = V(s) = I$. We know that they also satisfy the same differential system, since

$$U'(t) = \mathbf{\Phi}'(t)\mathbf{\Phi}^{-1}(s) = \mathbf{A}\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{A}U(t),$$
$$V'(t) = \mathbf{\Phi}'(t-s+t_0)\mathbf{\Phi}^{-1}(t_0) = \mathbf{A}\mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0) = \mathbf{A}V(t).$$

By the uniqueness of Theorem 1.6.1, the corresponding columns of U and V are identical since they satisfy the same equation with the same initial conditions. □

1.7 CONTINUATION OF SOLUTIONS

Theorem 1.7.1 *Assume that $\mathbf{X}(\mathbf{x}, t)$ is continuously differentiable in an open region $\mathcal{R}$ of $(\mathbf{x}, t)$-space. For any point $(\mathbf{c}, a) \in \mathcal{R}$, the initial value problem (1.4) has a unique solution $\mathbf{x}(t)$ defined over an interval $a \le t < b$ (b is a finite number or infinite) such that if $b < +\infty$, either $\mathbf{x}(t)$ is unbounded as $t \to b-$ or $(\mathbf{x}(t), t)$ approaches the boundary of the region $\mathcal{R}$.*[6]

Proof By the Local Existence Theorem 1.5.1, we know that there exist solutions of the initial value problem (1.4) in some interval $[a, T)$. Given two solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$, defined in $[a, T_1)$ and $[a, T_2)$ respectively, we define a new function $\mathbf{z}(t)$ to be either $\mathbf{x}(t)$ or $\mathbf{y}(t)$ wherever either is defined. Then $\mathbf{z}(t)$ is again a solution of (1.4) defined on $[a, \max\{T_1, T_2\})$. Thus, we can denote

$$b = \sup\left\{T \;\middle|\; \text{the initial value problem (1.4) has a solution in } [a, T)\right\}.$$

[6]The conclusion of Theorem 1.7.1 indicates that there are only three possible outcomes

1. $b = +\infty$ or
2. $b < +\infty$ and $\mathbf{x}(t)$ is unbounded as $t \to b-$ or
3. $b < +\infty$ and $(\mathbf{x}(t), t)$ approaches the boundary of $\mathcal{R}$.

We can define a single solution, denoted again by $\mathbf{x}(t)$, called the *maximal solution*, defined in $[a, b)$. The above construction indicates the existence of the maximal solution. It is also unique, by Uniqueness Theorem 1.2.1.

Let us consider the limiting behavior of $\mathbf{x}(t)$ as $t \to b-$. Then there are only possibilities:

Case 1: $b = +\infty$.

Case 2: $b < +\infty$ and $\mathbf{x}(t)$ is unbounded as $t \to b-$.

Case 3: $b < +\infty$ and $\mathbf{x}(t)$ is bounded as $t \to b-$.

We only need to show that in the third case, $\mathbf{x}(t)$ approaches the boundary of the region $\mathcal{R}$ as $t \to b-$. Indeed, let $\{t_n\}$ be any sequence such that $t_n \to b-$. Since the sequence of points $\{(\mathbf{x}(t_n), t_n)\}$ is bounded, there exists at least one limit point, say $(\mathbf{d}, b)$. We now show that the point $(\mathbf{d}, b)$ is on the boundary of $\mathcal{R}$. In fact, if it is an interior point, then there exists a closed neighborhood $D : \|\mathbf{x} - \mathbf{d}\| \le \epsilon, |t - b| \le \epsilon$ of $(\mathbf{d}, b)$ also in $\mathcal{R}$. Let $M = \max_D \|\mathbf{X}(\mathbf{x}(t), t)|\,\|$. Take $\delta < \min\{\epsilon, \epsilon/2M\}$ and let $G \subset D$ be the open set

$$G = \left\{(\mathbf{x}, , t)\middle|\|\mathbf{x} - \mathbf{d}\| < \epsilon, |t - b| < \delta\right\}.$$

We can take k large enough such that $(\mathbf{x}(t_k), t_k) \in G$. Applying Local Existence Theorem 1.5.1 to the equation $\dfrac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{X}(\mathbf{y}(t), t)$, we know that there exists a unique solution in the interval $|t - b| < \delta$ satisfying $\mathbf{y}(t_k) = \mathbf{x}(t_k)$. If we define

$$\mathbf{z}(t) = \begin{cases} \mathbf{x}(t), & a \le t < b, \\ \mathbf{y}(t), & b \le t < b + \delta, \end{cases}$$

then clearly $\mathbf{z}(t)$ is a solution of the initial value problem (1.4) over the interval $[a, b + \delta)$, contradicting to the maximality of b. □

Example 1.7.1 *Find the maximal solutions for the following initial value problems:*

$$(1) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x, \\ x(0) = c; \end{cases} \qquad (2) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2, \\ x(0) = c; \end{cases}$$

$$(3) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{t^2} \cos \dfrac{1}{t}, \\ x(t_0) = c, \quad t_0 \neq 0. \end{cases}$$

Solution

(1) The function $X(x, t) = x$ is continuously differentiable at any point $(x, t) \in (-\infty, +\infty) \times (-\infty, +\infty)$. The differential system has a unique solution $x(t) = ce^t$, which is the maximal solution defined for $t \in [0, +\infty)$.

(2) The function $X(x, t) = x^2$ is continuously differentiable at any point $(x, t) \in (-\infty, +\infty) \times (-\infty, +\infty)$. If $c = 0$, the maximal solution is $x(t) = 0$, defined for $t \in [0, +\infty)$. If $c \neq 0$, the maximal solution is $x(t) = \dfrac{1}{c - t}$, which is defined for $t \in [0, +\infty)$ if $c < 0$; or $t \in [0, c)$ if $c > 0$. Obviously, in the later case, $x(t)$ becomes unbounded as $t \to c-$.

(3) The function $X(x, t) = \dfrac{1}{t^2} \cos \dfrac{1}{t}$ is continuously differentiable at any point $(x, t) \in (-\infty, +\infty) \times (-\infty, +\infty)$ with $t \neq 0$. The maximal solution is $x(t) = c - \sin \dfrac{1}{t} + \sin \dfrac{1}{t_0}$, which is defined for $t \in [t_0, 0)$ if $t_0 < 0$; or for $t \in [t_0, +\infty)$ if $t_0 > 0$. In the former case, the function $x(t)$ is still bounded, but $t = 0$ is the boundary of

$$\mathcal{R} = \left\{(x, t) \;\middle|\; t \neq 0\right\}.$$

□

1.8 MISCELLANEOUS PROBLEMS

1. Show that the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 + y - \cos s, & \dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2 + \sin s, \\ x(0) = s + 1, & y(0) = s^2/(s^2 - 1) \end{cases}$$

has a unique solution $(x_s(t), y_s(t))$ in $|t| \leq T$, $|s| \leq 1/2$ for some T. Prove

$$\lim_{s \to 0} (x_s(t), y_s(t)) = (1, 0).$$

Solution For any $K_1 > 0$ and $T_1 > 0$, in the closed bounded region

$$\mathcal{R} = \left\{ (x, y, t, s) \middle| \sqrt{x^2 + y^2} \leq K_1, |t| \leq T_1, |s| \leq 1/2 \right\}$$

the function $\left(x^2 + y - \cos s, y^2 + \sin s\right)$ is continuously differentiable for all its variables (x, y, t, s). Thus, we know that it is a Lipschitz function in this region. By the Local Existence Theorem, for any fixed s with $|s| \leq 1/2$, the given initial value problem has a unque solution for $|t| \leq T_s = \min\{T_1, K_1/M_s\}$, where

$$M_s = \sup_{\substack{\sqrt{x^2+y^2} \leq K_1 \\ |t| \leq T_1}} \sqrt{[x^2 + y - \cos s]^2 + [y^2 + \sin s]^2}.$$

It is obvious that

$$M = \max_{|s| \leq 1/2} M_s < +\infty.$$

Thus, for any s with $|s| \leq 1/2$, the given initial value problem has a unque solution for $|t| \leq T = \min\{T_1, K_1/M\}$.

Denote the Lipschitz constant of the function $\left(x^2 + y + t - \cos s, y^2 + \sin s\right)$ in $\mathcal{R}$ to be L. It is easy to see that

$$\left\|(x^2 + y - \cos s, y^2 + \sin s) - (x^2 + y - 1, y^2)\right\| = \|(1 - \cos s, \sin s)\|.$$

By the Strong Continuity Theorem, the solutions $(x_s(t), y_s(t))$ and $(x_0(t), y_0(t)) = (1, 0)$, of the following initial value problems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 + y - \cos s, & \dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2 + \sin s, \\ x(0) = s + 1, & y(0) = s^2/(s^2 - 1), \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 + y - 1, & \dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2, \\ x(0) = 1, & y(0) = 0, \end{cases}$$

respectively, satisfy

$$\begin{aligned} & \|(x_s(t), y_s(t)) - (1,0)\| \\ \leq \; & \left\|(s+1, s^2/(s^2-1)) - (1,0)\right\| e^{L|t|} + \frac{1}{L}\,\|(1-\cos s, \sin s)\| \left(e^{L|t|} - 1\right). \end{aligned}$$

Let $s \to 0$, since the right hand side of the last inequality approaches zero, we have

$$\lim_{s \to 0} (x_s(t), y_s(t) = (1,0).$$

□

2. Use the Local Existence Theorem to show that the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 1 + x^2, \\ x(0) = 0, \end{cases}$$

has a unique solution for $|t| \leq 1/2$. Determine the region where the true solution is defined by solving this initial value problem. What is the limiting behavior of the true solution as t approaches the end point(s) of the maximal interval of existence?

Solution Consider the given initial value problem in the region

$$\left\{ (x,t) \;\middle|\; |x| \leq 1, |t| \leq 1 \right\}.$$

The function $X(x,t) = 1 + x^2$ satisfies the following Lipschitz condition

$$|X(x,t) - X(y,t)| = |x+y| \cdot |x-y| \leq 2|x-y|$$

for $|x| \leq 1$ and $|y| \leq 1$. Hence, by the Local Existence Theorem, there exists a unique solution of the given initial value problem for $|t| \leq \min\{1, 1/M\}$, where

$$M = \sup_{\substack{|x| \leq 1 \\ |t| \leq 1}} \|X(x,t)\| = \sup_{\substack{|x| \leq 1 \\ |t| \leq 1}} \left(1 + x^2\right) = 2.$$

That is, there exists a unique solution for $|t| \leq 1/2$.

The unique true solution of the given initial value problem is

$$x(t) = \tan t,$$

whose maximal interval of existence is $(-\pi/2, \pi/2)$. As $t \to \pm\pi/2$, $x(t)$ becomes unbounded. □

3. Assume that the function $f(x,t)$ is continuously differentiable in

$$\mathcal{R} = (-\infty, +\infty) \times (a,b),$$

and satisfies the inequality

$$|f(x,t)| \leq A(t)|x| + B(t),$$

for some non-negative continuous functions $A(t)$ and $B(t)$. Show that any solution of

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), \\ x(t_0) = c, \qquad t_0 \in (a,b), \end{cases}$$

has a maximal interval of existence (a,b).

Proof Let $x = x(t)$ be a solution of the initial value problem . We only show that it can be extend to the interval $[t_0, b)$. The continuation of the solution to $(a, x_0]$ can be proved similarly.

We prove it by contradiction. Suppose that the maximal interval of existence is $[t_0, \beta)$, with $\beta < b$. Select t_1 and t_2 such that

$$t_0 < t_1 < \beta < t_2 < b \text{ and } t_2 - t_1 < t_1 - t_0.$$

Denote $T = t_2 - t_1 > 0$. Let A_M and B_M be positive upper-bounds of $A(t)$ and $B(t)$ in the interval $[t_0, t_2]$, respectively. Thus, by the condition, we have

$$|f(x,t)| \le A_M|x| + B_M,$$

for $(x,t) \in (-\infty, +\infty) \times [t_0, t_2]$. We assume A_M is large enough such that $T < \dfrac{1}{A_M}$. Now we will see that the solution $x = x(t)$ can be extended to the interval $[t_0, t_2)$, a contradiction.

In fact, since $t_1 \in (t_0, \beta)$ and the solution $x = x(t)$ exists on $[t_0, \beta)$, for any positive number K, the region

$$\mathcal{R}_1 = \left\{(x,t)\Big| |x - x(t_1)| \le K, |t - t_1| \le T\right\}$$

is a bounded closed subset of $\mathcal{R}$. In $\mathcal{R}_1$, since

$$|f(x,t)| \le A_M|x| + B_M \le A_M(|x(t_1)| + K) + B_M = M,$$

by the Local Existence Theorem, the solution curve $(x(t), t)$ exists and remains in the region

$$\mathcal{R}_2 = \left\{(x,t)\Big| |x - x(t_1)| \le K, |t - t_1| \le h\right\},$$

where

$$h = \min\{T, K/M\}.$$

Since $\mathcal{R}_2$ is a bounded closed region, by Theorem 1.7.1, the solution curve $(x(t), t)$ can be extended to the boundary of $\mathcal{R}_2$. That is, the solution exists in $[t_0, t_1 + h)$. Since

$$\lim_{K \to +\infty} \frac{K}{M} = \frac{1}{A_M} > T,$$

we know that for sufficient large K,

$$h = \min\{T, K/M\} = T.$$

Thus, the solution $x = x(t)$ can be extended to $[t_0, t_1 + T) = [t_0, t_2)$.

This contradiction implies $\beta = b$. □

2

Plane Autonomous Systems

2.1 PLANE AUTONOMOUS SYSTEMS

Recall a normal form of a system of first-order ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} &=& X_1(x_1, \ldots, x_n; t), \\ & \vdots & \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} &=& X_n(x_1, \ldots, x_n; t). \end{cases}$$

If $n = 2$ and the functions X_i are independent of t, we have a so-called plane autonomous system. In this case, we write the autonomous system as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Y(x, y). \tag{2.1}$$

Let $(x(t), y(t))$ be a solution of system (2.1), we get a curve in $\mathbb{R}^2$, the phase plane, which is called the *solution curve* of the system. The differential system gives the tangent vector of the solution curve, $(X(x(t), y(t)), Y(x(t), y(t)))$. A $\mathbb{R}^2$ plane together with all solution curves is called the *phase plane* of the differential system.

The orientation of a solution curve is the direction of the movement of points on the curve when t increases. For example, the solution curve $\left(e^{2t}, e^t\right)$ of the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y, \end{cases}$$

is a parabola in Fig. 2.1(a). The solution curve $\left(e^{-2t}, e^{-t}\right)$ of the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -y, \end{cases}$$

is the same parabola in Fig. 2.1(b), with the opposite orientation.

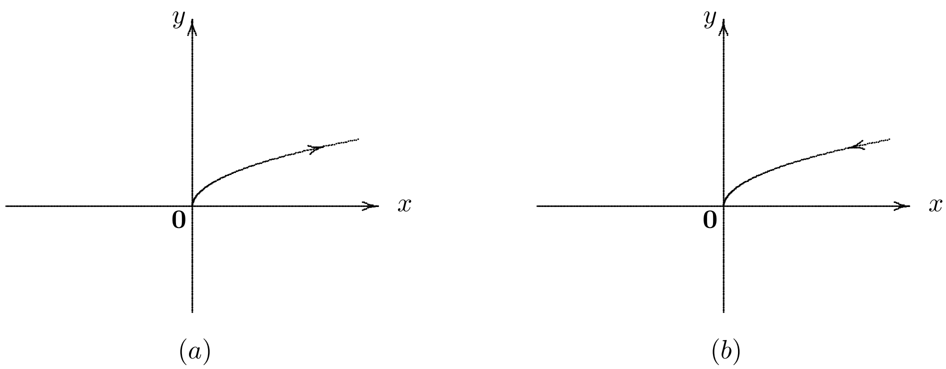


Fig. 2.1 Solution curves with orientations

Example 2.1.1 *Consider the differential system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y, \end{cases}$$

Sketch its oriented solutions curves in (x, y)-plane.

Solution The general solution is given by $(x(t), y(t)) = (c_1 e^{2t}, c_2 e^t)$, where c_1 and c_2 are arbitrary constants. For different values of c_1 and c_2, we have one solution curve. They are all shown in Fig. 2.2. □

In Example 2.1.1, the point $(0, 0)$ is a solution curve of the differential system even though it is not a curve in the ordinary sense. In Fig. 2.2, the point $(0, 0)$ is a special point, at which the direction is indeterminate. Such points are of particular importance for the study of autonomous systems.

Definition 2.1.1 *A point (x_0, y_0) is called a critical point of the autonomous system (2.1), if $X(x_0, y_0) = y(x_0, y_0) = 0$.*

If a point is not a critical point, we call it an *ordinary point*. Locally, the solution curves near an ordinary point are a family of parallel lines. However, a local structure near a critical point can be very complicated. If (x_1, y_1) is a neighboring point of a critical point (x_0, y_0), the solution curve

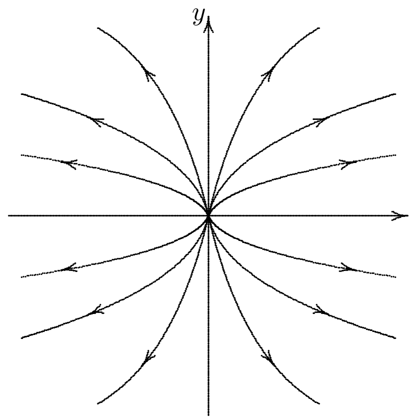


Fig. 2.2 Solution curves with orientations

passing through (x_1, y_1) can either stay close to the solution $(x(t), y(t)) \equiv (x_0, y_0)$, or move away from it, as t increases. In the former case, we say the critical point (x_0, y_0) is stable; in the later case, it is unstable. In a rigorous mathematical language, we introduce the following definition.

Definition 2.1.2 *Let $\mathbf{x}^0 = (x_0, y_0)$ be a critical point of the autonomous system (2.1), The point (x_0, y_0) is called*

(i) stable when, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$ for all $t > 0$ and all solution $\mathbf{x}(t) = (x(t), y(t))$ of the system (2.1) satisfying $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$;

(ii) attractive when, for some $\delta > 0$, $\lim\limits_{t \to +\infty} \|\mathbf{x}(t) - \mathbf{x}_0\| = 0$ for all solutions $\mathbf{x}(t) = (x(t), y(t))$ of the system (2.1) satisfying $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$;

(iii) strictly stable when it is stable and attractive;

(iv) neutrally stable when it is stable but not attractive;

(v) unstable when it is not stable.

For linear homogeneous autonomous systems, we will see that an attractive critical point must be a strictly stable point, but in general, this is not true. An example given by T. Brown demonstrates this. The phase diagram is given in Fig. 2.3. Obviously, for any solution $\mathbf{x}(t) = (x(t), y(t))$, the critical point $(0, 0)$ is attractive since $\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0$. However, it is not stable (so not strictly stable), since for any $\epsilon > 0$, no matter how small a neighborhood of the origin is, we can find an initial point in the neighborhood such that $\|\mathbf{x}(t)\| \geq \epsilon$ for some t.

Theorem 2.1.3 *Let $V(x, y)$ be a continuously differentiable function. Consider the plane autonomous system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V_y(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -V_x(x, y). \tag{2.2}$$

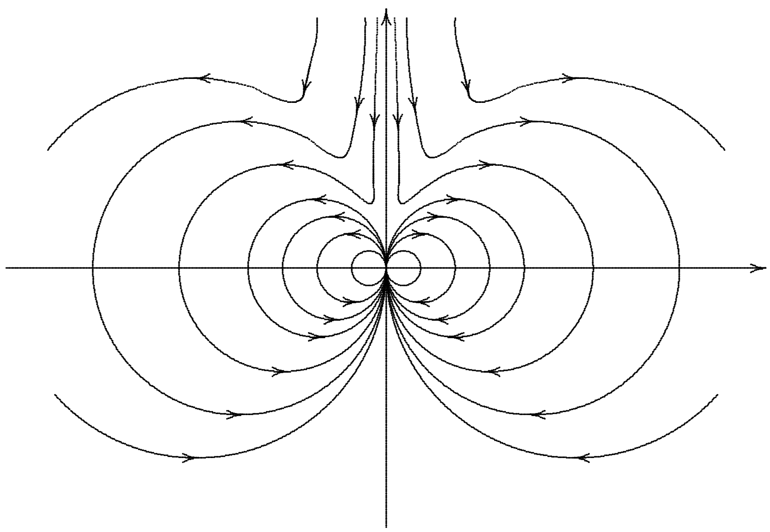


Fig. 2.3 The origin is attractive but not stable

Then each solution curve of the system lies on some level curve $V(x, y) = C$, where C is a constant.

Proof Let $(x(t), y(t))$ be a solution curve in (x, y)-plane. Then

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(x(t), y(t)) &= \frac{\partial V}{\partial x}(x(t), y(t)) \cdot \frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial V}{\partial y}(x(t), y(t)) \cdot \frac{\mathrm{d}y}{\mathrm{d}t}(t) \\ &= \frac{\partial V}{\partial x}(x(t), y(t)) \cdot \frac{\partial V}{\partial y}(x(t), y(t)) + \frac{\partial V}{\partial y}(x(t), y(t)) \cdot (-1)\frac{\partial V}{\partial x}(x(t), y(t)) \\ &= 0.\end{aligned}$$

Hence $V(x(t), y(t)) =$ constant. That is, the solution curve $(x(t), y(t))$ lies on some level curve $V(x, y) = C$. □

It is easy to show that an autonomous system (2.1) is in the form of (2.2) if and only if

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0,$$

provided both $X(x, y)$ and $Y(x, y)$ are continuously differentiable.

Example 2.1.2 *Sketch the solution curves in the phase plane for the equation* $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = x^3 - x$.

Solution The equation can be re-written as an autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x^3 - x.$$

It is easy to check that for $X(x, y) = y$ and $Y(x, y) = x^3 - x$,

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0 + 0 = 0.$$

To find the function $V(x, y)$ as in (2.2), we set

$$V_y(x, y) = y, \qquad V_x = -(x^3 - x).$$

Solving the first equation in the last system gives $V(x, y) = \frac{1}{2}y^2 + h(x)$, with h being an arbitrary function in x. Substituting the expression of V into the second equation, we have $h'(x) = -(x^3 - x)$. Solving for h, we obtain

$$V(x, y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2 + C,$$

with C a arbitrary constant. Now, by Theorem 2.1.3, we know that each solution curve of the differential system lies on some level curve

$$\frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2 = C,$$

where C is a constant. For different values of C, we sketch the level curves as in Fig. 2.4, which gives the phase diagram of solution curves. Notice that there exist three critical points: $(0, 0)$, (-10) and $(1, 0)$. It can be shown that $(0, 0)$ is neutrally stable and the others are unstable. □

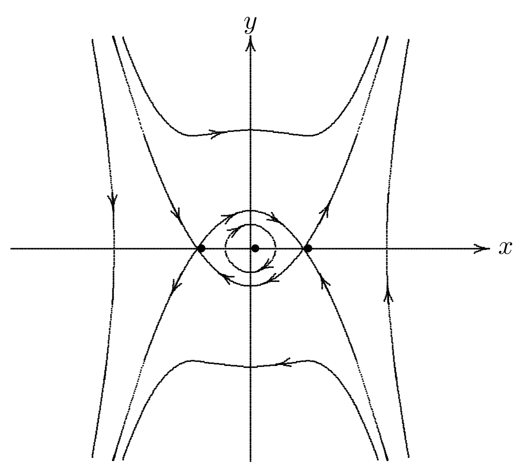


Fig. 2.4 Solution curves with three critical points

Example 2.1.3 *Sketch the solution curves in the phase plane for the equation*

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -k^2 \sin\theta, \qquad k^2 = g/l$$

which governs the motion of a simple pendulum of length l.

Solution The equation of a simple pendulum can be re-written as an autonomous system:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = v, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -k^2 \sin\theta.$$

If we write $X(\theta, v) = v$ and $Y(\theta, v) = -k^2 \sin\theta$, then

$$\frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial v} = 0 + 0 = 0.$$

Thus, we can write the autonomous system as in (2.2). Indeed, for

$$V_v(\theta, v) = v, \qquad V_\theta(\theta, v) = k^2 \sin\theta,$$

we solve the first equation to have $V(\theta, v) = \frac{1}{2}v^2 + h(\theta)$. The second equations yields $h(\theta) = -k^2 \cos\theta + C$. Thus, we have $V(\theta, v) = \frac{1}{2}v^2 - k^2 \cos\theta + C$. By Theorem 2.1.3, we know that each solution curve of the differential system lies on some level curve

$$\frac{1}{2}v^2 - k^2 \cos\theta = C,$$

where C is a constant. In the phase plane, the (v, θ)-plane, the last equation indicates that if $(v(t), \theta(t))$ is a solution curve, so is $(v(t), \theta(t) + 2\pi)$. We sketch its solution curves in Fig. 2.5 for the case $k^2 = 1$. Obviously, there are infinitely many critical points at $(0, n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$. □

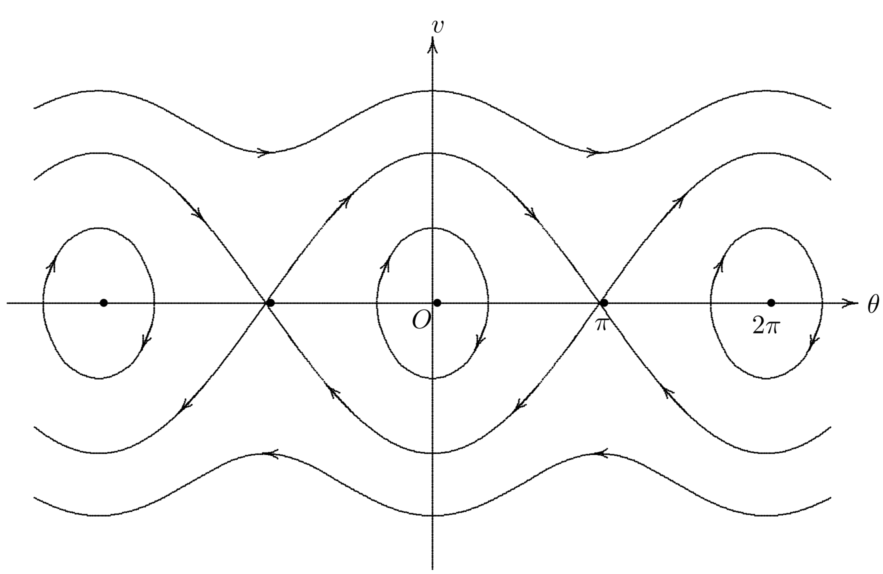


Fig. 2.5 Solution curves for a simple pendulum

2.2 LINEAR AUTONOMOUS SYSTEMS

In this section, we investigate two dimensional autonomous systems of the form

$$x' = ax + by, \qquad y' = cx + dy, \tag{2.3}$$

where a, b, c and d are real numbers. It can be written in the following vector form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4}$$

It is obvious that the origin $(0, 0)$ is a critical point of the system. If $\det \mathbf{A} = ad - bc \neq 0$, there are no other critical points. If $\det \mathbf{A} = 0$, then any nontrivial solution of the system $ax + by = 0$ and $cx + dy = 0$ is another critical point. The main purpose of this section is to make a complete classification for the critical point $(0, 0)$.

Theorem 2.2.1 *(1) Let ϕ_1 and ϕ_2 be two real linearly independent eigenvectors of the matrix $\mathbf{A}$, with λ_1 and λ_2 being the corresponding eigenvalues. Then a fundamental matrix of the homogeneous system (2.4) is given by*

$$\left(e^{\lambda_1 t}\phi_1, e^{\lambda_2 t}\phi_2\right).$$

(2) Let $\phi_1 = \phi_r + i\phi_i$ be a complex eigenvector, with $\lambda_1 = \lambda_r + i\lambda_i$ the corresponding eigenvalue ($\lambda_i \neq 0$). Then a fundamental matrix is given by

$$\left(e^{\lambda_r t}(\phi_r \cos\lambda_i t - \phi_i \sin\lambda_i t), e^{\lambda_r t}(\phi_i \cos\lambda_i t + \phi_r \sin\lambda_i t)\right).$$

(3) If $\lambda_1 = \lambda_2 = \lambda$ and ϕ is the only linearly independent eigenvector, then there exists another vector ψ, being linearly independent to ϕ and satisfying $(\mathbf{A} - \lambda\mathbf{I})\psi = \phi$, such that

$$\left(e^{\lambda t}\phi, te^{\lambda t}\phi + e^{\lambda t}\psi\right)$$

is a fundamental matrix.[1]

Proof (1) By Theorem 1.6.2, we know that

$$e^{\lambda_1 t}\phi_1, e^{\lambda_2 t}\phi_2$$

[1]When the eigenvalues are repeated and there is only one linearly independent eigenvector, the matrix $\mathbf{A}$ is not diagonalizable. Or equivalently, $\mathbf{A}$ is similar to the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In other words, there is an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\mathbf{P}.$$

are linearly independent, since ϕ_1, ϕ_2 are. It remains to prove both are solutions of (2.4). Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda_1 t}\phi_1\right) = e^{\lambda_1 t}\lambda_1\phi_1 = e^{\lambda_1 t}\mathbf{A}\phi_1 = \mathbf{A}\left(e^{\lambda_1 t}\phi_1\right),$$

i.e., $e^{\lambda_1 t}\phi_1$ is a solution. It can be proved for $e^{\lambda_2 t}\phi_2$ similarly.

(2) By (1), we know that $e^{(\lambda_r+i\lambda_i)t}(\phi_r+i\phi_i)$ is a solution of the homogeneous system (2.4). That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{(\lambda_r+i\lambda_i)t}(\phi_r+i\phi_i)\right) = \mathbf{A}\left(e^{(\lambda_r+i\lambda_i)t}(\phi_r+i\phi_i)\right).$$

Equating the real and imaginary parts respectively in the last equation, we have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda_r t}(\phi_r\cos\lambda_i t - \phi_i\sin\lambda_i t)\right) &= \mathbf{A}\left(e^{\lambda_r t}(\phi_r\cos\lambda_i t - \phi_i\sin\lambda_i t)\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda_r t}(\phi_i\cos\lambda_i t + \phi_r\sin\lambda_i t)\right) &= \mathbf{A}\left(e^{\lambda_r t}(\phi_i\cos\lambda_i t + \phi_r\sin\lambda_i t)\right). \end{cases}$$

Thus, $e^{\lambda_r t}(\phi_r\cos\lambda_i t - \phi_i\sin\lambda_i t)$ and $e^{\lambda_r t}(\phi_i\cos\lambda_i t + \phi_r\sin\lambda_i t)$ are solutions. To show they are linearly independent, by Theorem 1.6.2, we only need to show that they are linealy independent at $t=0$. In other words, it is sufficient to show that ϕ_r and ϕ_i are linearly independent.

Assume that there are constants c_1 and c_2 such that

$$c_1\phi_r + c_2\phi_i = 0. \tag{2.5}$$

Since

$$\begin{cases} \mathbf{A}\phi_r &= \lambda_r\phi_r - \lambda_i\phi_i, \\ \mathbf{A}\phi_i &= \lambda_i\phi_r + \lambda_r\phi_i, \end{cases}$$

which can be obtained by equating the real and imaginary parts of $A(\phi_r+i\phi_i) = (\lambda_r+i\lambda_i)(\phi_r+i\phi_i)$, multiplying $\mathbf{A}$ to both sides of equation (2.5), we obtain

$$(\lambda_r c_1 + \lambda_i c_2)\phi_r + (-\lambda_i c_1 + \lambda_r c_2)\phi_i = 0.$$

Multiplying $-\lambda_r$ to equation (2.5) and adding it to the last equation, we have

$$\lambda_i c_2\phi_r - \lambda_i c_1\phi_i = 0.$$

Since $\lambda_i \neq 0$, the last equation gives

$$c_2\phi_r - c_1\phi_i = 0.$$

Combining this equation with (2.5), we can easily have

$$(c_1^2+c_2^2)\phi_r = (c_1^2+c_2^2)\phi_i = 0.$$

These imply $c_1 = c_2 = 0$. That is, ϕ_r and ϕ_i are linearly independent.

(3) Let us first show the existence of such vector ψ. Since $\lambda_1=\lambda_2=\lambda$ and λ satisfies

$$\lambda^2 - (a+d)\lambda + (a\ d - b\ c) = 0,$$

we know that

$$(a-d)^2 + 4b\ c = 0 \text{ and } \lambda = \frac{a+d}{2}.$$

These imply

$$\begin{aligned}(\mathbf{A}-\lambda\mathbf{I})^2 &= \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}^2 \\ &= \begin{pmatrix} (a-\lambda)^2+b\,c & b(d-\lambda)+b(a-\lambda) \\ c(a-\lambda)+c(d-\lambda) & (d-\lambda)^2+b\,c \end{pmatrix} = 0.\end{aligned}$$

Since ϕ is the only linearly independent eigenvector corresponding to the eigenvalue λ, there must be a vector ψ such that $(\mathbf{A}-\lambda\mathbf{I})\psi \neq 0$. But $(\mathbf{A}-\lambda\mathbf{I})^2\psi = 0$ implies that there exists a non-zero constant c such that $(\mathbf{A}-\lambda\mathbf{I})\psi = c\cdot\phi$. Dividing by c and renaming $\frac{1}{c}\cdot\psi$ to be ψ, we have

$$(\mathbf{A}-\lambda\mathbf{I})\psi = \phi.$$

The linear independence of ϕ and ψ can be shown as follows. Assume

$$\alpha\phi + \beta\psi = 0.$$

Multiplying $\mathbf{A}-\lambda\mathbf{I}$ to the left of the last equation, we have $\beta\phi = 0$, which implies $\beta = 0$. This can be used further to get $\alpha = 0$.

By (1), we know that $e^{\lambda t}\phi$ is a solution of (2.4). By a direct verification:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(te^{\lambda t}\phi + e^{\lambda t}\psi\right) &= e^{\lambda t}\phi + \lambda te^{\lambda t}\phi + \lambda e^{\lambda t}\psi \\ &= te^{\lambda t}(\lambda\phi) + e^{\lambda t}\left(\lambda\psi + \phi\right) \\ &= te^{\lambda t}\mathbf{A}\phi + e^{\lambda t}\mathbf{A}\psi \\ &= \mathbf{A}\left(te^{\lambda t}\phi + e^{\lambda t}\psi\right),\end{aligned}$$

we know that $te^{\lambda t}\phi + e^{\lambda t}\psi$ is also a solution of (2.4). By Theorem 1.6.2, these two solutions are linearly independent, since at $t = 0$, they are ϕ_1, ϕ_2 which are linearly independent. □

It is convenient to introduce the following concept.

Definition 2.2.2 *Two first-order linear homogeneous autonomous systems* $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ *and* $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{B}\mathbf{u}$ *are said linearly equivalent if there exists a non-singular matrix* $\mathbf{K}$ *such that* $\mathbf{x}(t) = \mathbf{K}\mathbf{u}(t)$.

It can be seen easily that if two autonomous systems are linearly equivalent, then the coefficient matrices are similar. In fact, since

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{K}^{-1}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{K}^{-1}\mathbf{A}\mathbf{x} = (\mathbf{K}^{-1}\mathbf{A}\mathbf{K})\mathbf{u},$$

we conclude that

$$\mathbf{B} = \mathbf{K}^{-1}\mathbf{A}\mathbf{K},$$

i.e., $\mathbf{A}$ and $\mathbf{B}$ are similar.

Two linearly equivalent autonomous systems have similar structures at the critical point $(0, 0)$, since $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ indicates that solution curves only undergo “stretching” and “rotating” transformations.

Theorem 2.2.3 *The origin is a stable (attractive) point for differential system* $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ *if and only if it is a stable (attractive) point for a linearly equivalent differential system* $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{B}\mathbf{u}$.

Proof By symmetry, it suffices to show that if $(0,0)$ is stable (attrative) for $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$, then it is stable (attractive) for $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{B}\mathbf{u}$.

Since $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ and $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{B}\mathbf{u}$ are linearly equivalent, there is a non-singular matrix $\mathbf{P}$, such that

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \mathbf{P} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Thus, for any t, we have

$$\begin{aligned} \|\mathbf{u}(t)\| &= \sqrt{u^2(t) + v^2(t)} \\ &= \sqrt{(p_{11}x + p_{12}y)^2 + (p_{21}x + p_{22}y)^2} \\ &= \sqrt{(p_{11}^2 + p_{21}^2)x^2 + (p_{12}^2 + p_{22}^2)y^2 + 2(p_{11}p_{12} + p_{21}p_{22})xy} \\ &\le \sqrt{(p_{11}^2 + p_{21}^2)x^2 + (p_{12}^2 + p_{22}^2)y^2 + |p_{11}p_{12} + p_{21}p_{22}|(x^2 + y^2)} \\ &\le M\sqrt{x^2 + y^2} = M\,\|\mathbf{x}(t)\|, \end{aligned}$$

where

$$M = \sqrt{2\max\{p_{11}^2 + p_{21}^2, p_{12}^2 + p_{22}^2, |p_{11}p_{12} + p_{21}p_{22}|\}}.$$

Similarly, since $\mathbf{x}(t) = \mathbf{P}^{-1}\mathbf{u}(t)$, there is a constant N such that

$$\|\mathbf{x}(t)\| \le N\,\|\mathbf{u}(t)\|.$$

Suppose $(0,0)$ is stable for the system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$. For any $\epsilon > 0$, there exists $\delta_1 > 0$ such that if $\|\mathbf{x}(0)\| < \delta_1$, we have $\|\mathbf{x}(t)\| < \epsilon/M$ for all $t > 0$. Take $\delta = \delta_1/N$. If $\|\mathbf{u}(0)\| < \delta$, then $\|\mathbf{x}(0)\| \le N\|\mathbf{u}(0)\| < N\delta = \delta_1$. Hence $\|\mathbf{u}(t)\| \le M\|\mathbf{x}(t)\| < M \cdot \epsilon/M = \epsilon$. Therefore $(0,0)$ is stable for the system $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{B}\mathbf{u}$.

Suppose $(0,0)$ is attractive for the system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}$. Then there exists a δ_1 such that whenever $\|\mathbf{x}(0)\| < \delta_1$, we have $\lim_{t\to+\infty} \|\mathbf{x}(t)\| = 0$. Take $\delta = \delta_1/N$. If $\|\mathbf{u}(0)\| < \delta$, we have $\|\mathbf{x}(0)\| \le N\|\mathbf{u}(0)\| < N\delta = \delta_1$. Hence we have

$$0 \le \lim_{t\to+\infty} \|\mathbf{u}(t)\| \le \lim_{t\to+\infty} M\|\mathbf{x}(t)\| = 0,$$

which implies $\lim_{t\to+\infty} \|\mathbf{u}(t)\| = 0$. Hence $(0,0)$ is attrative for the system $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = B\mathbf{u}$ as well. □

By Theorem 2.2.3 and Definition 2.1.2, we know that linear equivalence does not change the qualitative nature of the phase portrait for linear systems. For instance, if the origin is a neutrally stable point for a linear differential system, then it will be a neutrally stable point for all its linearly equivalent linear differential systems.

2.3 COMPLETE CLASSIFICATION FOR LINEAR AUTONOMOUS SYSTEMS

Theorem 2.3.1 *For different cases, we have the following complete classification:*

(1) If **A** *has two linearly independent real eigenvectors, with λ_1, λ_2 being the eigenvalues, then (2.4) is linearly equivalent to the canonical form:*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \lambda_1 u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \lambda_2 v. \end{cases} \tag{2.6}$$

(a) If λ_1 and λ_2 are both negative, then the origin is a stable nodal point, as in Fig. 2.6(a). Furthermore, if $\lambda_1 = \lambda_2$, then the origin is a stable star point as in Fig. 2.7(a).

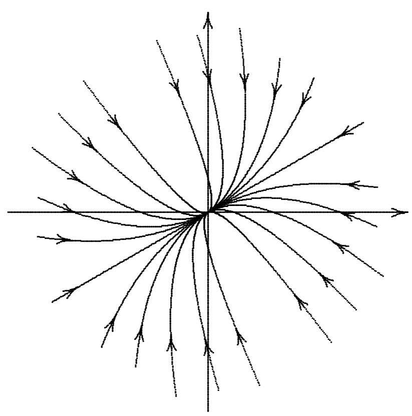
(b) If λ_1 and λ_2 are both positive, then the origin is an unstable nodal point, as in Fig. 2.6(b). Furthermore, if $\lambda_1 = \lambda_2$, then the origin is an unstable star point as in Fig. 2.7(b).

(c) If λ_1 and λ_2 are of opposite sign, then the origin is an unstable saddle point, as in Fig. 2.8.

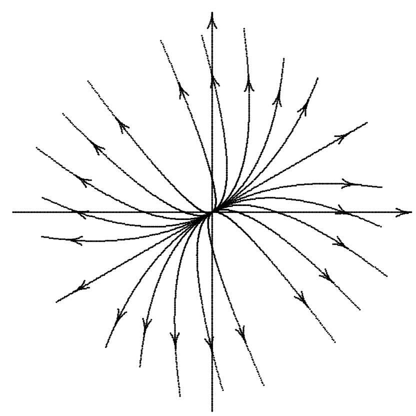
(d) If $\lambda_1 < 0$ and $\lambda_2 = 0$, then the origin is neutrally stable, as in Fig. 2.9(d).

(e) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then the origin is unstable, as in Fig. 2.9(e).

(f) If $\lambda_1 = \lambda_2 = 0$, then the origin is neutrally stable, as in Fig. 2.10. In this case, **A** *is the zero matrix.*



Case (a): stable nodal point if $\lambda_1, \lambda_2 < 0$

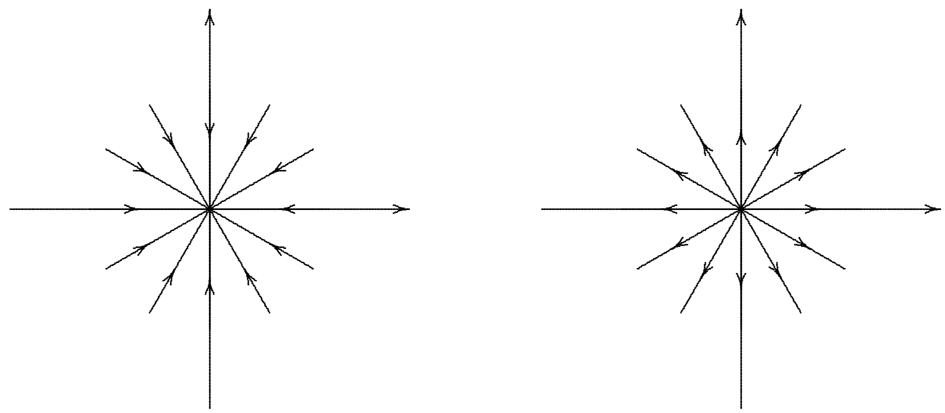


Case (b): unstable nodal point if $\lambda_1, \lambda_2 > 0$

Fig. 2.6 **A** has two linearly independent real eigenvectors and λ_1 and λ_2 are distinct and of the same sign

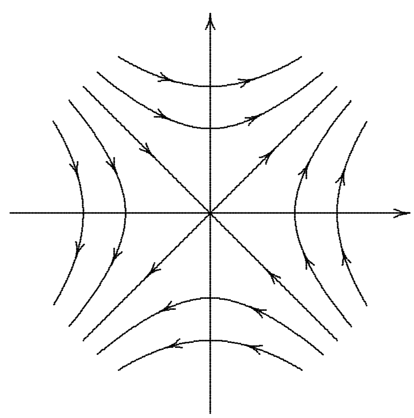
(2) If **A** *has two linearly independent complex eigenvectors, with $\lambda_1, \lambda_2 = \lambda_r \pm i\lambda_i$ being the eigenvalues, then (2.4) is linearly equivalent to the canonical form:*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \lambda_r u - \lambda_i v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \lambda_i u + \lambda_r v. \end{cases} \tag{2.7}$$



Case (a): stable star point if $\lambda_1 = \lambda_2 < 0$ Case (b): unstable star point if $\lambda_1 = \lambda_2 > 0$

Fig. 2.7 **A** has two linearly independent eigenvector but repeated eigenvalues



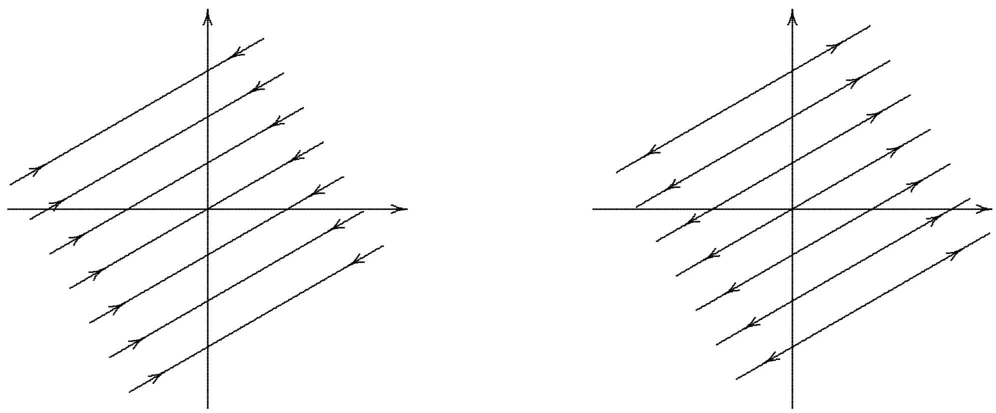
Case (c): unstable saddle point if λ_1 and λ_2 are of opposite sign

Fig. 2.8 **A** has two linearly independent real eigenvectors and λ_1 and λ_2 are of opposite sign

(a) If $\lambda_r < 0$, then the origin is a stable focal point, as in Fig. 2.11(a).

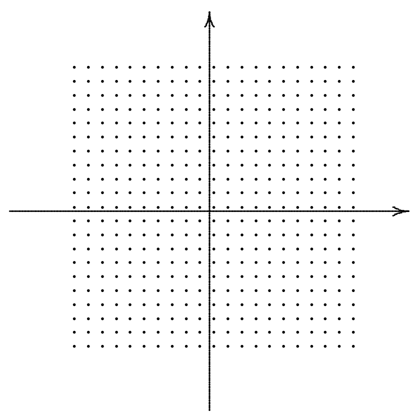
(b) If $\lambda_r > 0$, then the origin is an unstable focal point, as in Fig. 2.11(b).

(c) If $\lambda_r = 0$, then the origin is a stable vortex point, as in Fig. 2.12.



Case (d): neutrally stable if $\lambda_1 < 0$ and $\lambda_2 = 0$ Case (e): neutrally unstable if $\lambda_1 > 0$ and $\lambda_2 = 0$

Fig. 2.9 **A** has two linearly independent real eigenvectors and one of the eigenvalues is zero



Case (f): neutrally stable point if $\lambda_1 = 0$ and $\lambda_2 = 0$

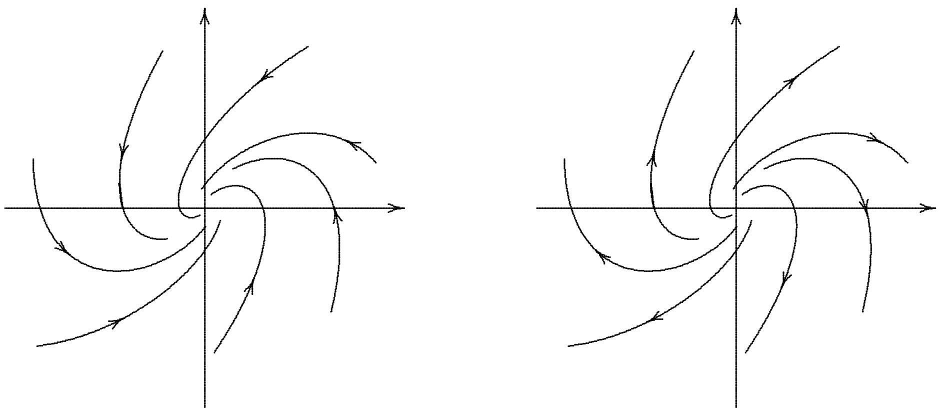
Fig. 2.10 $\mathbf{A} = \mathbf{0}$

(3) If **A** *has only one repeated eigenvalue* λ *and if there exists only one linearly independent eigenvector, then (2.4) is linearly equivalent to the canonical form:*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \lambda u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = u + \lambda v. \end{cases} \tag{2.8}$$

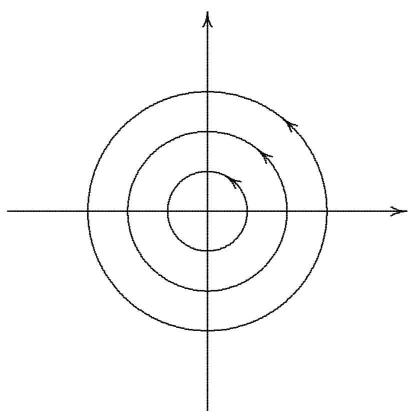
(a) If $\lambda < 0$*, then the origin is a stable nodal point, as in Fig. 2.13(a).*

(b) If $\lambda > 0$*, then the origin is an unstable nodal point, as in Fig. 2.13(b).*



Case (a): stable focal point if $\lambda_r < 0$ Case (b): unstable focal point if $\lambda_r > 0$

Fig. 2.11 **A** has two linearly independent complex eigenvectors



Case (c): neutrally stable vortex point if $\lambda_r = 0$

Fig. 2.12 **A** has two linearly independent complex eigenvectors

(c) If $\lambda = 0$, then the origin is unstable, as in Fig. 2.14.

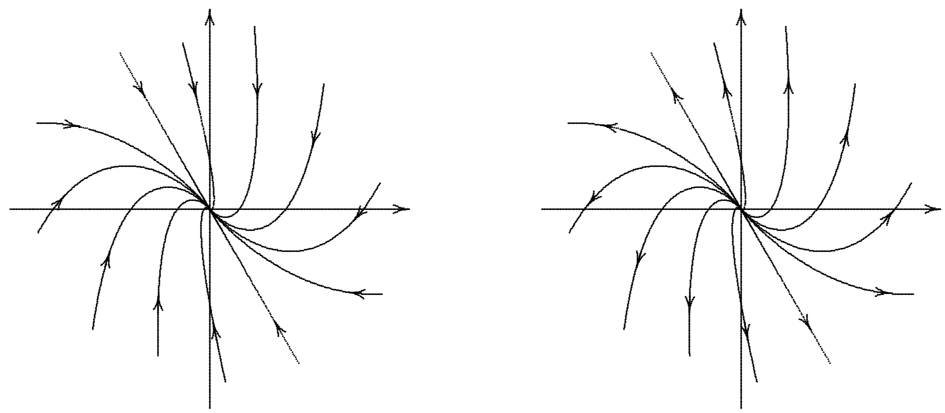
Proof

(1) Assume that ϕ_1 and ϕ_2 are two linearly independent real eigenvectors. Take $\mathbf{K} = (\phi_1, \phi_2)$ and denote

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{K}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{K}^{-1}\mathbf{x}.$$

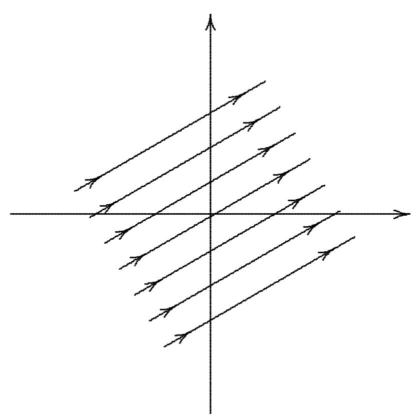
Since

$$\mathbf{AK} = (\mathbf{A}\phi_1, \mathbf{A}\phi_2) = (\lambda_1\phi_1, \lambda_2\phi_2) = \mathbf{K} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$



Case (a): stable nodal point if $\lambda < 0$ Case (b): unstable nodal point if $\lambda > 0$

Fig. 2.13 **A** has only one linearly independent eigenvector (repeated eigenvalues) and $\lambda \neq 0$



Case (c): unstable point if $\lambda = 0$

Fig. 2.14 **A** has only one linearly independent eigenvector (repeated eigenvalues) and $\lambda = 0$

we have

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{K}^{-1}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{K}^{-1}\mathbf{A}\mathbf{x} = \mathbf{K}^{-1}\mathbf{A}\mathbf{K}\mathbf{u} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{u}.$$

This is the desired canonical form.[2]

[2]The identity

$$\mathbf{A} = \mathbf{K}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{K}^{-1}.$$

(1a) If λ_1 and λ_2 are both negative, by solve the canonical equation, we have $(u(t), v(t)) = \left(ae^{\lambda_1 t}, be^{\lambda_2 t}\right)$. Thus,

$$u = cv^m, \qquad m = \lambda_2/\lambda_1 > 0.$$

In (u, v)-plane, this is a family of parabola-like curves. Since as $t \to +\infty$, $(u(t), v(t)) \to (0, 0)$, we know the directions are towards the origin. Together with the linear transformation $\mathbf{K}$, which generally gives a stretching and/or rotating for the solution curves, we have the schematic figure of solution curves in (x, y)-plane, as in Fig. 2.6(a).

(1b) In the case of both λ_1, λ_2 being positive, it is the same as in (1a), except that we have to switch the orientations of the solution curves, as in Fig .2.6(b).

A special case in (1a) and (1b) is that $\lambda_1 = \lambda_2$. We have $u = cv$, a family of straight lines passing through the origin. Correspondingly, as in Fig. 2.7, the phase diagram is a so-called star: the origin is stable if $\lambda < 0$; unstable if $\lambda < 0$.

(1c) If λ_1 and λ_2 are of opposite sign, then similarly we have

$$uv^m = c, \qquad m = -\lambda_2/\lambda_1 > 0.$$

For different values of c, this gives a family of hyperbola-like curves, as in Fig 2.8. Hence, in this case, we have a saddle point. As $t \to +\infty$, either $u(t)$ or $v(t)$ approaches $+\infty$. Thus, a saddle point is always unstable.

(1d) In this case, the solution of the canonical equation is $(u(t), v(t)) = \left(ae^{\lambda_1 t}, b\right)$. Thus, we have a family of straight lines, since for fixed b, the solution curve always has $v(t) = b$. Since as $t \to +\infty$, $(u(t), v(t)) \to (0, b)$, we conclude that the origin is stable but not strictly stable, as in Fig. 2.9(d).

(1e) This is the case similar to (1d), except the orientation is opposite. Thus, the origin is unstable, as in Fig. 2.9(e).

(1f) In the case of $\lambda_1 = \lambda_2 = 0$, the solution of the canonical equation degenerates to a single point $(u(t), v(t)) = (a, b)$. The origin is stable but not strictly stable, as in Fig. 2.10.

(2) Assume $\phi_r + i\phi_i$ is an eigenvector corresponding to the eigenvalue $\lambda_r + i\lambda_i$. In the proof of Theorem 2.2.1, we know

$$\mathbf{A}\phi_r = \lambda_r\phi_r - \lambda_i\phi_i, \qquad \mathbf{A}\phi_i = \lambda_i\phi_r + \lambda_r\phi_i,$$

It is also proved there that ϕ_r and ϕ_i are linearly independent. Take $\mathbf{K} = (\phi_r, -\phi_i)$ and denote $\mathbf{u} = \mathbf{K}^{-1}\mathbf{x}$. Since

$$\mathbf{AK} = (\mathbf{A}\phi_r, -\mathbf{A}\phi_i) = (\lambda_r\phi_r - \lambda_i\phi_i, -\lambda_i\phi_r - \lambda_r\phi_i) = \mathbf{K}\begin{pmatrix} \lambda_r & -\lambda_i \\ \lambda_i & \lambda_r \end{pmatrix},$$

similarly we have

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{K}^{-1}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{K}^{-1}\mathbf{A}\mathbf{x} = \mathbf{K}^{-1}\mathbf{AKu} = \begin{pmatrix} \lambda_r & -\lambda_i \\ \lambda_i & \lambda_r \end{pmatrix}\mathbf{u}.$$

This is the desired canonical form.

(2a) Using the canonical form, we have

$$\frac{\mathrm{d}(u^2 + v^2)}{\mathrm{d}t} = 2u\frac{\mathrm{d}u}{\mathrm{d}t} + 2v\frac{\mathrm{d}u}{\mathrm{d}t} = 2u(\lambda_r u - \lambda_i v) + 2v(\lambda_i u + \lambda_r v) = 2\lambda_r(u^2 + v^2),$$

indicates that $\mathbf{A}$ is similar to this diagonal matrix: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ which is called the Jordan canonical form.

In fact, $\mathbf{P} = (\phi_1, \phi_2)$, formed by the linearly independent eigenvectors.

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\arctan\frac{v}{u}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{u}\right)}{1+\left(\frac{v}{u}\right)^2} = \frac{\frac{\mathrm{d}v}{\mathrm{d}t}u - \frac{\mathrm{d}u}{\mathrm{d}t}v}{u^2+v^2} = \frac{(\lambda_i u + \lambda_r v)u - (\lambda_r u - \lambda_i v)v}{u^2+v^2} = \lambda_i.$$

Solve these two equations, in the polar coordinates, we have

$$r = (u^2+v^2)^{1/2} = \rho e^{\lambda_r t}, \qquad \theta = \arctan\frac{v}{u} = \lambda_i t + \tau,$$

where $\rho \geq 0$ and τ are arbitrary constants. Thus, the phase diagram consists of contracting spirals surrounding the origin if $\lambda_r < 0$, as shown in Fig 2.11(a). Thus, the origin is a stable focal point.

(2b) If $\lambda_r > 0$, by the discussion in (2a), the phase diagram consists of expanding spirals from the origin, as shown in Fig 2.11(b). The origin is an unstable focal point.

(2c) If $\lambda_r = 0$, then by the discussion in (2a), we have a family of closed curves. In this case, the origin is a neutrally stable vortex point, or sometimes it is called a center, as in Fig. 2.12.

(3) Assume ϕ is the only linearly independent eigenvector corresponding to the eigenvalue λ. By Theorem 2.2.1, there is a vector ψ, linearly independent to ϕ, such that $\mathbf{A}\psi = \phi + \lambda\psi$. Take $\mathbf{K} = (\phi, \psi)$ and denote $\mathbf{u} = \mathbf{K}^{-1}\mathbf{x}$. Since

$$\mathbf{AK} = (\mathbf{A}\phi, \mathbf{A}\psi) = (\lambda\phi, \phi + \lambda\psi) = \mathbf{K}\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix},$$

similarly we have

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{K}^{-1}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{K}^{-1}\mathbf{Ax} = \mathbf{K}^{-1}\mathbf{AKu} = \begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}\mathbf{u}.$$

We obtain the desired canonical form.

(3a) The solution of the canonical equation is $(u(t), v(t)) = \left(ae^{\lambda t}, ate^{\lambda t} + be^{\lambda t}\right)$, where a and b are arbitrary constants. Thus, if $\lambda < 0$, we know that the origin is a stable nodal point, as in Fig 2.13(a).

(3b) If $\lambda > 0$, as in (3a), we know that the origin is an unstable nodal point, as in Fig. 2.13(b).

(3c) if $\lambda = 0$, the solution of the canonical equations degenerates to $(u(t), v(t)) = (a, at + b)$, which is a family of straight lines, as in Fig 2.14. This is a case that the origin is unstable. □

By Theorem 2.3.1 and Theorem 2.2.3, we immediately have the following results.

Corollary 2.3.2 *The critical point* $(0, 0)$ *of a linear homogeneous differential system* $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{Ax}$ *is attractive if and only if every eigenvalue of* $\mathbf{A}$ *has a negative real part. In this case, the origin is strictly stable.*

Example 2.3.1 *Consider the system* $x' = -x + 3y$ *and* $y' = y$*. Find the canonical form of this system such that both are linearly equivalent.*

Solution The coefficient matrix

$$\mathbf{A} = \begin{pmatrix}-1 & 3\\ 0 & 1\end{pmatrix},$$

which has two distinct real eigenvalues $\lambda_1 = -1, \lambda_2 = 1$. The corresponding eigenvectors

$$\phi_1 = \begin{pmatrix}1\\ 0\end{pmatrix}, \qquad \phi_2 = \begin{pmatrix}3\\ 2\end{pmatrix}$$

Take

$$\mathbf{K} = (\phi_1, \phi_2) = \begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}$$

and define

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{K}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -\frac{3}{2}x + \frac{1}{2}y \end{pmatrix}.$$

Then the differential system for (u, v) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{K}^{-1}\mathbf{AK}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

which is a canonical form. □

Example 2.3.2 *Consider the system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & \alpha x + 2y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} & = & -2x. \end{cases}$$

where α is a constant. Find the critical values of α where the change of the qualitative nature of the phase portrait for the system occurs. Classify the autonomous system.

Solution The eigenvalues λ satisfy the characteristic equation

$$\det\begin{pmatrix} \alpha-\lambda & 2 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 - \alpha\lambda + 4 = 0,$$

whose solutions are

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 16}}{2}, \qquad \lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 16}}{2}.$$

We consider the following three different cases:

(1) $\alpha^2 - 16 < 0$;

(2) $\alpha^2 - 16 = 0$;

(1) $\alpha^2 - 16 > 0$.

In the case (1), we have $|\alpha| < 4$ and the coefficient matrix has two complex eigenvalues, with the real part being $\frac{\alpha}{2}$. By Theorem 2.3.1, for $-4 < \alpha < 0$, the origin is a stable nodal point; for $0 < \alpha < 4$, the origin is an unstable nodal point; at $\alpha = 0$, the origin is a stable vortex.

In the case (2), for both $\alpha = 4$ and $\alpha = -4$, the eigenvalue is repeated and there exists only one eigenvector. By Theorem 2.3.1, the origin is an unstable critical point.

In the case (3), we have $|\alpha| > 4$ and the eigenvalues are real, distinct and of the same sign. Their signs are the same as that of α. Thus, by Theorem 2.3.1, the origin is a stable focal point if $-\infty < \alpha < -4$; an unstable focal point if $4 < \alpha < +\infty$. □

2.4 LIAPUNOV DIRECT METHOD

The classification of critical points of linear homogeneous autonomous differential systems in the previous section provides us a complete understanding on the local structures of such systems. However, local structures in the phase space for general differential systems are too complicated and it is beyond the scope of this course.

We may try to generalize the idea of so-called "linear equivalence". As we knew, a linear equivalence is a simple "stretching" and "rotating" transformation. This will not change the nature of local structures, as shown in Theorem 2.2.3. Obviously, for the purpose of preserving local structures, there are other transformations. In the phase space, we can stretch the solution curves a bit in some regions while squeeze in some other regions. We can also twist a little bit of the solutions curves. Such "nonlinear equivalence" is called *diffeomorphism*. However, finding a diffeomorphism between two systems is not trivial. Actually, it is practically useless to classify critical points for nonlinear autonomous systems in terms of diffeomorphisms.

One practically useful tool in the study local structures for nonlinear autonomous systems is Liapunov functions. Let us first use the following examples to demonstrate the basic ideas.

Example 2.4.1 *Show that the critical point* $(0,0)$ *of the plane autonomous system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &=& y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &=& -x-\mu(x^2-1)y, \end{cases}$$

is stable for any $\mu < 0$.

Solution Let $(x(t), y(t))$ be a solution of the system for $t \geq 0$. Consider $V(x,y) = x^2 + y^2$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(x(t),y(t)) &= 2x(t)\cdot\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2y(t)\cdot\frac{\mathrm{d}y(t)}{\mathrm{d}t} \\ &= 2x(t)\cdot y(t) + 2y(t)\cdot\big(-x(t)-\mu(x^2(t)-1)y(t)\big) \\ &= -2\mu(x^2(t)-1)y^2(t) \leq 0, \end{aligned}$$

if $|x(t)| \leq 1$. That is, the function $V(x(t),y(t))$ is a decreasing function along any solution curve with $|x(t)| \leq 1$. For any given $\epsilon > 0$, we choose $\delta = \min\{1, \epsilon\}$. Thus, if $\sqrt{x^2(0)+y^2(0)} < \delta$, then

$$x^2(t)+y^2(t) = V(x(t),y(t)) \leq V(x(0),y(0)) = x^2(0)+y^2(0) \leq \delta^2 \leq \epsilon^2.$$

In other words,

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon.$$

Therefore, the origin is a stable point of the system. □

It can be shown that, in fact, the origin is a strictly stable in the previous example. This will be seen later.

Example 2.4.2 *Show that the critical point* $(0,0)$ *of the regular plane autonomous system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &=& y-x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &=& -2(x^3+y^3), \end{cases}$$

is stable.

Solution Let $(x(t), y(t))$ be a solution of the system for $t \geq 0$. Choose $V(x,y) = x^4 + y^2$. Consider $V(t) = x(t)^4 + y(t)^2$.

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 4x(t)^3x'(t) + 2y(t)y'(t) = -4(x(t)^6 + y(t)^4) \leq 0.$$

Hence $V(t)$ is a decreasing function. Therefore when $t > 0$, $V(t) \le V(0)$.

Given $\epsilon > 0$, take $\delta = \min\{1, \dfrac{\epsilon^2}{\sqrt{2}}\}$. Whenever $\sqrt{x(0)^2 + y(0)^2} < \delta$, we have

$$x(t)^4 + y(t)^2 \le x(0)^4 + y(0)^2 \le x(0)^2 + y(0)^2 < \delta^2$$

since $x(0)^2 \le \delta^2 < 1$. Now we have

$$\begin{aligned}(x(t)^2 + y(t)^2)^2 &= x(t)^4 + y(t)^4 + 2x(t)^2y(t)^2 \\ &\le 2(x(t)^4 + y(t)^4) \\ &\le 2(x(t)^4 + y(t)^2) \le 2\delta^2 \le \epsilon^4\end{aligned}$$

where we used $|y(t)| < \delta < 1$. Hence $\sqrt{x(t)^2 + y(t)^2} < \epsilon$. Thus $(0, 0)$ is a stable point. □

The above examples indicates a new approach to study stability of nonlinear autonomous systems.

Before we generalize the idea of the above examples, we should clarify one point. Since existence and uniqueness are not always guaranteed for nonlinear systems, we have to make sure that the solutions for the discussion of stability do exist. It is convenient to introduce the following definition.

Definition 2.4.1 *A plane autonomous system (2.1) is said to be regular if for any initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$*, there exists a unique solution in the neighborhood of* $(\mathbf{x}_0, t_0)$*.*

By Theorem 1.5.1, we know that if $\mathbf{X}(\mathbf{x})$ is continuously differentiable for all $\mathbf{x}$, then the nonlinear system is regular.

In the two examples above, the existence of the function $V(x, y)$ is the central ingredient of the method. Let $(x(t), y(t))$ be a solution of (2.1). For a given function $V(x, y)$, denote

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t), y(t)) = V_x(x(t), y(t))X(x(t), y(t)) + V_y(x(t), y(t))Y(x(t), y(t))$$

to be the *total derivative* of V with respect to (2.1).

We introduce the following definition.

Definition 2.4.2 *A function* $V(x, y)$*, defined in a neighborhood of the origin, is called a weak Liapunov function for the plane sutonomous system (2.1) if*

(1) $V(x, y)$ *is continuously differentiable and* $V(0, 0) = 0$.

(2) $V(x, y)$ *is positive definite, in other words,* $V(x, y) > 0$ *if* $(x, y) \ne (0, 0)$.

(3)
$$\frac{\partial}{\partial x}V(x, y) \cdot X(x, y) + \frac{\partial}{\partial y}V(x, y) \cdot Y(x, y) \le 0,$$

for $(x, y) \ne (0, 0)$.

It will be called a strong Liapunov function if the above condition (3) is changed to

(3)′
$$\frac{\partial}{\partial x}V(x, y) \cdot X(x, y) + \frac{\partial}{\partial y}V(x, y) \cdot Y(x, y) < 0,$$

for $(x, y) \neq (0, 0)$.

Theorem 2.4.3 *If there exists a weak Liapunov function for the regular plane autonomous system (2.1), then the critical point* $(0, 0)$ *is stable. If there exists a strong Liapunov function, then the critical point* $(0, 0)$ *is strictly stable.*

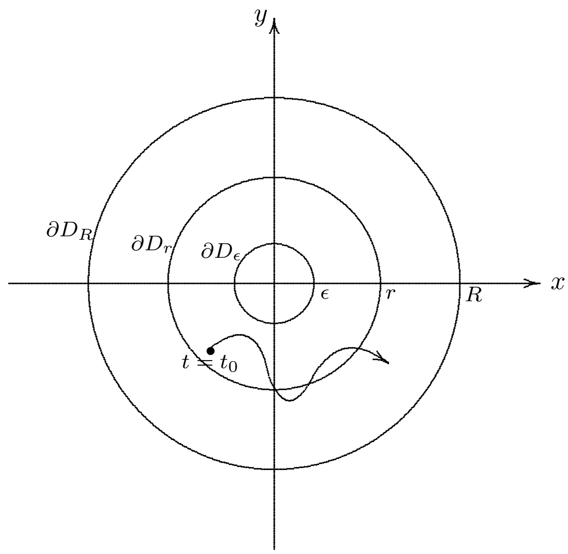


Fig. 2.15 A typical solution curve near the origin

Proof For each constant $a > 0$, we denote D_a the circular open disc of radius $a > 0$ centered at the origin. For any fixed $R > 0$, since the weak Liapunov function V is continuous on the boundary ∂D_R, which is a bounded closed set, V has a positive minimum m on ∂D_R. Since V is continuous and $V(0, 0) = 0$, there is a positive number $r < R$ such that $V(x, y) < m$ for $(x, y) \in D_r$. Let $(x(t), y(t))$ be any solution of (2.1) and $(x(t_0), y(t_0)) \in D_r$. Since $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), y(t)) \leq 0$, we know that

$$V(x(t), y(t)) \leq V(x(t_0), y(t_0)) < m, \qquad \text{for } t \geq t_0.$$

This implies that the solution curve can never reach the boundary ∂D_R. Hence, the origin is stable.

To show that the origin is strictly stable if V is a strong Liapunov function, we only need to show that $\lim_{t \to +\infty} V(x(t), y(t)) = 0$. Since $V(x, y)$ is positive definite and $V(0, 0) = 0$, this limit implies that $(x(t), y(t)) \to (0, 0)$, or the origin is strictly stable. In fact, since

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), y(t)) = \frac{\partial}{\partial x} V(x(t), y(t)) \cdot X(x(t), y(t)) + \frac{\partial}{\partial y} V(x(t), y(t)) \cdot Y(x(t), y(t)) < 0,$$

we know that $V(x(t), y(t))$ is decreasing in t. The condition that $V(x(t), y(t))$ is bounded below by 0 implies that the limit $\lim_{t \to +\infty} V(x(t), y(t))$ exists, say L. If $L > 0$, since $V(0, 0) = 0$, we take a small $\epsilon > 0$ is so that $V(x,y) < L$ in D_ϵ. In the ring regin bounded by ∂D_ϵ and ∂D_R, the function

$$\frac{\partial}{\partial x}V(x,y)\cdot X(x,y) + \frac{\partial}{\partial y}V(x,y)\cdot Y(x,y)$$

has a negative maximum $-k$, for some $k > 0$. Since the solution curve $(x(t), y(t))$ remains in this ring region, we have

$$V(x(t),y(t)) - V(x(t_0),y(t_0)) = \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}V(x(t),y(t))\mathrm{d}t \le -k(t-t_0),$$

for $t \ge t_0$. This inequality implies $V(x(t),y(t)) \to -\infty$, which contradicts to the fact $V(x(t),y(t)) \ge 0$. Thus, we conclude $L = 0$. □

Theorem 2.4.4 *If the critical point* $(0,0)$ *of the linear plane autonomous system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= ax+by, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= cx+dy, \end{cases}$$

is strictly stable, then so is that of the perturbed regular system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= ax+by+\xi(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= cx+dy+\eta(x,y), \end{cases} \tag{2.9}$$

provided there exist $k > 0$ *and* $M > 0$ *such that whenever* $x^2+y^2 < k$ *we have* $|\xi(x,y)| + |\eta(x,y)| \le M(x^2+y^2)$.

Proof For simplicity, since the linear homogeneous system is strictly stable at the origin, by Theorem 2.3.1, we can assume that it is one of the types

$$(1) \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= \lambda_1 x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= \lambda_2 y, \end{cases} \quad \lambda_1 < 0, \quad \lambda_2 < 0; \qquad (2) \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= \lambda_r x - \lambda_i y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= \lambda_i x + \lambda_r y, \end{cases} \quad \lambda_r < 0;$$

or

$$(3) \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= \lambda x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= x+\lambda y, \end{cases} \quad \lambda < 0.$$

Otherwise, we can transform the linear system to its canonical form via a linear equivalence transformation. The new perturbed system still satisfies the conditions in Theorem 2.4.4 after such transformation.

Take $V(x,y) = x^2+y^2$ for the first two cases above and $V(x,y) = x^2+\lambda^2y^2$ for the last case. To show that $V(x,y)$ is a strong Liapunov function, we only need to show that there exists a negative constant $\mu < 0$ such that

$$\frac{\partial}{\partial x}V(x,y)\cdot X(x,y) + \frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \le \mu V(x,y),$$

for sufficiently small (x,y).

In fact, in the case (1), we have

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \;& 2x\cdot(\lambda_1 x+\xi)+2y\cdot(\lambda_2 y+\eta) \\ \le \;& 2\lambda(x^2+y^2)+2(x\xi+y\eta), \end{aligned}$$

with $\lambda=\max\{\lambda_1,\lambda_2\}$. Similarly to the case (1), we can show that, for $x^2+y^2<\delta=\min\{k,\frac{\lambda^2}{16M^2}\}$, since $|x|\le-\frac{\lambda}{4M}$ and $|y|\le-\frac{\lambda}{4M}$, we have

$$|x\xi+y\eta|\le 2\left(\frac{-\lambda}{4M}\right)\cdot M(x^2+y^2)=-\frac{\lambda}{2}V(x,y),$$

which implies

$$\frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y)\le\lambda V(x,y),$$

whenever $x^2+y^2<\delta$.

In the case (2), we have

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \;& 2x\cdot(\lambda_r x-\lambda_i y+\xi)+2y\cdot(\lambda_r x+\lambda_i y+\eta) \\ = \;& 2\lambda_r(x^2+y^2)+2(x\xi+y\eta). \end{aligned}$$

By the condition,

$$|x\xi+y\eta|\le|x|\cdot|\xi|+|y|\cdot|\eta|\le(|x|+|y|)\cdot M(x^2+y^2),$$

whenever $x^2+y^2<k$. Thus, when $x^2+y^2<\delta=\min\{k,\frac{\lambda_r^2}{16M^2}\}$, since $|x|\le-\frac{\lambda_r}{4M}$ and $|y|\le-\frac{\lambda_r}{4M}$, we have

$$|x\xi+y\eta|\le 2\left(-\frac{\lambda_r}{4M}\right)\cdot M(x^2+y^2)=-\frac{\lambda_r}{2}(x^2+y^2).$$

Hence,

$$\frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y)\le 2\lambda_r(x^2+y^2)+(-\lambda_r)(x^2+y^2)=\lambda_r V(x,y),$$

whenever $x^2+y^2<\delta$.

In the case (3), $V(x,y)=x^2+\lambda^2y^2$ and

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \;& 2x\cdot(\lambda x+\xi)+2\lambda^2y\cdot(x+\lambda y+\eta) \\ = \;& \lambda(x^2+\lambda^2y^2)+\lambda(x+\lambda y)^2+2(x\xi+\lambda^2y\eta) \\ \le \;& \lambda(x^2+\lambda^2y^2)+2(x\xi+\lambda^2y\eta). \end{aligned}$$

By the condition,

$$|x\xi+\lambda^2y\eta|\le|x|\cdot|\xi|+\lambda^2|y|\cdot|\eta|\le(|x|+\lambda^2|y|)\cdot M(x^2+y^2),$$

whenever $x^2 + y^2 < k$. Thus, when $x^2 + y^2 < \delta = \min\{k, \frac{\lambda^2}{64M^2(1+\lambda^2)^2(1+\lambda^{-2})^2}\}$, since

$$|x| \leq \frac{-\lambda}{8M(1+\lambda^2)(1+\lambda^{-2})}, \quad |y| \leq \frac{-\lambda}{8M(1+\lambda^2)(1+\lambda^{-2})},$$

we have

$$\begin{aligned} |x\xi + \lambda^2 y\eta| &\leq (1+\lambda^2)(|x|+|y|)) \cdot M \cdot (1+\lambda^{-2})(x^2+\lambda^2 y^2) \\ &\leq (1+\lambda^2) \cdot 2 \cdot \frac{-\lambda}{8M(1+\lambda^2)(1+\lambda^{-2})} \cdot (1+\lambda^{-2})(x^2+\lambda^2 y^2) \\ &= \frac{-\lambda}{4}(x^2+\lambda^2 y^2). \end{aligned}$$

Hence,

$$\frac{\partial}{\partial x}V(x,y) \cdot X(x,y) + \frac{\partial}{\partial y}V(x,y) \cdot Y(x,y) \leq \lambda(x^2+\lambda^2 y^2) + \frac{-\lambda}{2}(x^2+\lambda^2 y^2) = \frac{\lambda}{2}V(x,y),$$

whenever $x^2 + y^2 < \delta$. □

Now we can apply Theorem 2.4.4 to Example 2.4.1. The system in Example 2.4.1 is perturbed from the linear system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + \mu y. \end{cases}$$

The origin is strictly stable for the linear system, since the coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

has the eigenvalues

$$\frac{\mu \pm \sqrt{\mu^2 - 4}}{2},$$

which both have negative real parts for $\mu < 0$.

Example 2.4.3 *Show that for the differential system*

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{1}{m}\{f(x,y)y + \lambda x\}, \end{cases}$$

the critical point $(0,0)$ *is stable, where* $m > 0$, $\lambda > 0$ *and* $f(x,y) \geq 0$ *in a neighborhood of the origin*

Solution In fact, take

$$V(x,y) = \frac{1}{2}(\lambda x^2 + my^2).$$

Then

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \quad & \lambda xy - my\frac{1}{m}\left[f(x,y)y+\lambda x\right] \\ = \quad & -my^2 f(x,y) \le 0. \end{aligned}$$

By Theorem 2.4.3, the origin is stable. □

Example 2.4.4 *Discuss the dependence on the sign of the constant μ of the critical point at the origin of the regular system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y+\mu x^5, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x+\mu y^5. \end{cases}$$

Solution When $\mu = 0$, the system becomes $x' = -y$ and $y' = x$ and the origin is a neutrally stable vortex point.

Suppose $\mu \neq 0$. Let $V(x,y) = x^2+y^2$. Then

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y)+\frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \quad & 2x(-y+\mu x^5)+2y(x+\mu y^5) \\ = \quad & 2\mu(x^6+y^6) < 0, \end{aligned}$$

for $(x,y)\neq(0,0)$, since $\mu<0$. Thus $V(x,y)$ is a strong Liapunov function. By Theorem 2.4.3, the origin is strictly stable.

If $\mu>0$, denote $V'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}V(x(t),y(t))$. Then the previous calculation gives $V'(t)\geq 0$. Hence $V(t)$ is increasing. Therefore $x(t)^2+y(t)^2 \geq x(0)^2+y(0)^2$ for $t>0$. Since

$$\begin{aligned} (x^2+y^2)^3 &= x^6+y^6+3x^4y^2+3x^2y^4 \\ &= x^6+y^6+3x^2y^2(x^2+y^2) \\ &\leq x^6+y^6+3(x^2+y^2)\Big(\frac{x^2+y^2}{2}\Big)^2 \\ &= x^6+y^6+\frac{3}{4}(x^2+y^2)^3, \end{aligned}$$

we conclude $x^6+y^6 \geq \dfrac{1}{4}(x^2+y^2)^3$. Now we have

$$V'(t) = 2\mu(x(t)^6+y(t)^6) \geq 2\mu\frac{1}{4}(x(t)^2+y(t)^2)^3 \geq \frac{\mu}{2}(x(0)^2+y(0)^2)^3.$$

Take $k = \dfrac{\mu}{2}(x(0)^2+y(0)^2)^3 > 0$. Consider the function $V(t)-kt$. Since $(V(t)-kt)' = V(t)'-k \geq 0$, hence $V(t)-kt$ is an increasing function. For $t>0$, we have $V(t)-kt \geq V(0) = x(0)^2+y(0)^2$. Therefore $\lim\limits_{t\to+\infty} V(t) \geq \lim\limits_{t\to+\infty}(kt+x(0)^2+y(0)^2) = +\infty$. Given $\epsilon = 1$, for any $\delta>0$, if $x(0)^2+y(0)^2<\delta$, since $\lim\limits_{t\to+\infty} V(t) = +\infty$, there exists N depending only upon $x(0)$ and $y(0)$ such that whenever $t>N$, $V(t) = x(t)^2+y(t)^2 \geq 1$. Therefore $(0,0)$ is unstable. □

We can generalize the idea in the second part of the last example to get a test for instability of a critical point. This will be discussed in the next section.

Exercise 2.4

1. Show that the function $ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$.

2. Show that the critical point is strictly stable for the following system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= -4x - y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -2x - 5y - 2y\sin x. \end{cases}$$

3. Show that for any $a < 0$, the critical point $(0,0)$ of the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= ax + y + x^2 y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -x + ay\cos x, \end{cases}$$

is strictly stable.

4. Show that the origin $(0,0)$ is strictly stable for the following system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= xy - 2x^2y^3 - x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -y - \dfrac{1}{2}x^2 + x^3y^2. \end{cases}$$

5. Show that the the origin $(0,0)$ is strictly stable for the following system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= -2(x^3 + y^3), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= x - 2y^3. \end{cases}$$

6. (1) Show that the critical point $(0,0)$ is stable to the following regular system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -2x - 3(1 + y^2)y \end{cases}$$

by constructing a weak Liapunov function.

(2) Show that the critical point $(0,0)$ is attractive to the system.

2.5 A TEST FOR INSTABILITY

Theorem 2.5.1 *Assume that there exists a function, $U(x,y)$ defined in $x^2+y^2 \le k^2$, a neighborhood of the origin, satisfying that*

(1) $U(x,y)$ is continuously differentiable and $U(0,0) = 0$;

(2)

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x(t), y(t)) > 0,$$

for any solution $(x(t), y(t))$ of the system (2.1) with $(x(t), y(t)) \neq (0,0)$;

(3) in every neighborhood of the origin, there exists at least one point (x, y) such that $U(x, y) > 0$.

Then the critical point $(0,0)$ is unstable for the regular plane autonomous system (2.1) .

Proof By the condition, for each $\delta > 0$, $0 < \delta < k$, there exists a point (x_δ, y_δ) such that $0 < x_\delta^2 + y_\delta^2 < \delta^2$ and $U(x_\delta, y_\delta) > 0$. Denote $(x_\delta(t), y_\delta(t))$, $t \geq t_0$, to be the solution curve satisfying $(x_\delta(t_0), y_\delta(t_0)) = (x_\delta, y_\delta)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}U(x_\delta(t), y_\delta(t)) > 0$, we know that $U(x_\delta(t), y_\delta(t)) \geq U(x_\delta, y_\delta) > 0$, for $t \geq t_0$. Thus the solution curve cannot enter the origin as $t \to +\infty$, since $U(0,0) = 0$. Let D_{ϵ_0} be the circular disc in which is solution curve $(x_\delta(t), y_\delta(t))$ is bounded away. Then in the ring region $D_k - D_{\epsilon_0}$, the function $\frac{\mathrm{d}}{\mathrm{d}t}U(x_\delta(t), y_\delta(t))$ has a positve minimum m:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x_\delta(t), y_\delta(t)) \geq m > 0,$$

for $t \geq t_0$. This implies

$$U(x_\delta(t), y_\delta(t)) - U(x_\delta, y_\delta) = \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}U(x_\delta(t), y_\delta(t))\mathrm{d}t \geq m(t - t_0).$$

Let $t \to +\infty$, we conclude that $U(x_\delta(t), y_\delta(t))$ is unbounded. Therefore the solution curve $(x_\delta(t), y_\delta(t))$ cannot remain in the disc D_k. So this solution curve must reach the boundary ∂D_k.

Therefore, for any fixed ϵ, $0 < \epsilon < k$, for any $\delta > 0$, there exists at least one solution $(x_\delta(t), y_\delta(t))$ satisfying

$$x_\delta^2(t_0) + y_\delta^2(t_0) < \delta^2 \text{ but } x_\delta^2(t) + y_\delta^2(t) > \epsilon^2 \text{ for some } t.$$

In other words, the origin is not a stable critical point. □

Theorem 2.5.2 *If the critical point $(0,0)$ of the linear plane autonomous system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= ax + by, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= cx + dy, \end{cases}$$

is unstable, where the eigenvalues of the coefficient matrix are different, non-zero, and at least one has positive real part, then the perturbed regular system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= ax + by + \xi(x, y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= cx + dy + \eta(x, y), \end{cases}$$

is unstable at the origin $(0,0)$, provided there exist $k > 0$ and $M > 0$ such that whenever $x^2 + y^2 < k$ we have $|\xi(x, y)| + |\eta(x, y)| \leq M(x^2 + y^2)$.

Proof Since the eigenvalues of the coefficient matrix of the linear system are different, non-zero, and at least one has positive real part, for simplicity, by Theorem 2.3.1, we can assume that it is one of the types

$$(1) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= \lambda_r x - \lambda_i y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= \lambda_i x + \lambda_r y, \end{cases} \quad \lambda_r > 0; \qquad (2) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= \lambda_1 x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= \lambda_2 y, \end{cases} \quad \lambda_1 > 0.$$

We will show that there exist functions $U(x,y)$ satisfying the conditions in Theorem 2.5.1, repectively.

In the case (1), consider the function $U(x,y) = x^2 + y^2$. We only need to show that the function U satisfies the condition (2) in Theorem 2.5.1.

Let $(x(t), y(t))$ be a solution of the perturbed system and write $U(t) = U(x(t), y(t))$. For any solution of the perturbed system, since

$$\begin{aligned} \frac{\mathrm{d}U(t)}{\mathrm{d}t} &= 2x(t)\cdot\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2y(t)\cdot\frac{\mathrm{d}y(t)}{\mathrm{d}t} \\ &= 2x\cdot(\lambda_r x - \lambda_i y + \xi) + 2y\cdot(\lambda_i x + \lambda_r y + \eta) \\ &= 2\lambda_r(x^2+y^2) + 2(x\xi + y\eta), \end{aligned}$$

By the condition,

$$|x\xi + y\eta| \le |x|\cdot|\xi| + |y|\cdot|\eta| \le (|x|+|y|)\cdot M(x^2+y^2),$$

whenever $x^2 + y^2 < k$. Thus, when $x^2 + y^2 < \tau = \min\{k, \frac{\lambda_r^2}{16M^2}\}$, since $|x| \le \frac{\lambda_r}{4M}$ and $|y| \le \frac{\lambda_r}{4M}$, we have

$$|x\xi + y\eta| \le 2\left(\frac{\lambda_r}{4M}\right)\cdot M(x^2+y^2) = \frac{\lambda_r}{2}(x^2+y^2).$$

Hence,

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} \ge 2\lambda_r(x^2+y^2) + (-\lambda_r)(x^2+y^2) = \lambda_r U(t),$$

whenever $x^2(t) + y^2(t) < \tau$.

In the case (2), consider the function $U(x,y) = x^2/\lambda_1 + y^2/\lambda_2$. Since $\lambda_1 > 0$, this function obviously satisfies the condition (3) in Theorem 2.5.1. Again, we only need to show that the function U satisfies the condition (2) in Theorem 2.5.1.

For any solution $(x(t), y(t))$ of the perturbed system,

$$\begin{aligned} \frac{\mathrm{d}U(t)}{\mathrm{d}t} &= 2x(t)\cdot\frac{\mathrm{d}x(t)}{\mathrm{d}t}/\lambda_1 + 2y(t)\cdot\frac{\mathrm{d}y(t)}{\mathrm{d}t}/\lambda_2 \\ &= 2x\cdot(\lambda_1 x + \xi)/\lambda_1 + 2y\cdot(\lambda_2 y + \eta)/\lambda_2 \\ &= 2(x^2+y^2) + 2(x\xi/\lambda_1 + y\eta/\lambda_2), \end{aligned}$$

By the condition,

$$|x\xi/\lambda_1 + y\eta/\lambda_2| \le |x|\cdot|\xi|/|\lambda_1| + |y|\cdot|\eta|/|\lambda_2| \le (|x|+|y|)(\min\{|\lambda_1|,|\lambda_2|\})^{-1}\cdot M(x^2+y^2),$$

whenever $x^2 + y^2 < k$. Thus, when $x^2 + y^2 < \tau = \min\{k, \frac{\min\{\lambda_1^2,\lambda_2^2\}}{16M^2}\}$, since $|x| \le \frac{\min\{|\lambda_1|,|\lambda_2|\}}{4M}$ and $|y| \le \frac{\min\{|\lambda_1|,|\lambda_2|\}}{4M}$, we have

$$|x\xi/\lambda_1 + y\eta/\lambda_2| \le 2\left(\frac{\min\{|\lambda_1|,|\lambda_2|\}}{4M}\right)(\min\{|\lambda_1|,|\lambda_2|\})^{-1}\cdot M(x^2+y^2) = \frac{1}{2}(x^2+y^2).$$

Hence,

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} \ge 2(x^2+y^2) + (-1)(x^2+y^2) = x^2(t) + y^2(t),$$

whenever $x^2(t) + y^2(t) < \tau$. □

Example 2.5.1 *Determine the stability of the critical points of*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & y(x+1), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} & = & x(1+y^3). \end{cases}$$

Solution There are two critical points for this system: $(0,0)$ and $(-1,-1)$. At $(0,0)$, the linearization is

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} & = & x. \end{cases}$$

The eigenvalues of the coefficient matrix are ± 1. Hence the origin is a saddle point for the linearized system. By Theorem 2.5.2, the origin is unstable.

Near the other critical point $(-1,-1)$, we shift it to the origin by making

$$x = -1 + u, \qquad y = -1 + v.$$

Then (u,v) satisfies the following system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} & = & (v-1)u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} & = & (u-1)\left[1+(v-1)^3\right]. \end{cases}$$

The linearization at $(0,0)$ of this system is

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} & = & -u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} & = & -3v, \end{cases}$$

whose eigenvalues of the coefficient matrix are -1 and -3. By Theorem 2.4.4, this perturbed nonlinear system is strictly stable at $(0,0)$. So the original system is strictly stable at $(-1,-1)$. □

Exercise 2.5

1. Show that for any constants $a > 0$ and $b > 0$, the origin is an unstable critical point for the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & -x^2y + ax^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} & = & 2x^3 + by^7. \end{cases}$$

2. Let μ be a real number. Discuss the stability of the critical point $(0,0)$ for the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & -y + \mu x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} & = & x + \mu y^3. \end{cases}$$

3. Assume that $f(x,y)$ is continuously differentiable. For the system

$$\begin{cases} \dfrac{dx}{dt} &= y - xf(x,y), \\ \dfrac{dy}{dt} &= -x - yf(x,y), \end{cases}$$

show that

(a) the origin $(0,0)$ is strictly stable if $f(x,y) > 0$ in a neughborhood of the origin;

(b) the origin $(0,0)$ is unstable if $f(x,y) < 0$ in a neughborhood of the origin.

2.6 NONLINEAR OSCILLATIONS

In mechanics, differential equations of the form

$$\ddot{x} + p(x,\dot{x})\dot{x} + q(x) = 0 \tag{2.10}$$

often appear in many different occasions. Generally, they describe the displacement x of a particle of unit mass under a force system containing a conservative element, $q(x)$, and a dissipative or energy-generating component, $p(x,\dot{x})$. In this section, we consider the stability for two special but common cases of this type of nonlinear oscillators.

2.6.1 Undamped Oscillations

When $p(x,\dot{x}) = 0$, the nonlinear oscillator (2.10) is said to be *undamped*. Then the above equation is equivalent to the plane autonomous system

$$\begin{cases} \dfrac{dx}{dt} &= v, \\ \dfrac{dv}{dt} &= -q(x). \end{cases} \tag{2.11}$$

We will study stability about a critical point $(x_0, 0)$, where $q(x_0) = 0$. Without loss of generality, we may assume $x_0 = 0$, otherwise a translation of coordinates can move $(x_0, 0)$ to $(0,0)$. If the origin $(0,0)$ is stable, we expect that the "restoring force" $q(x)$ must act in the opposite direction to the displacement x in a neighborhood of $x = 0$, that is, $xq(x) > 0$ for $x \neq 0$ sufficiently small.

Theorem 2.6.1 *If $q \in \mathcal{C}^1$ and if $xq(x) > 0$ for small nonzero x, then the critical point $(0,0)$ of the system $\ddot{x} + q(x) = 0$ is a vortex point.*

Proof We define the potential energy integral:

$$V(x) = \int_0^x q(\xi)d\xi.$$

For any given positive constant E, the locus $v^2/2 + V(x) = E$ is an integral curve, since we differentiate it to yield an identity: $\dot{E} = \dot{x}[\ddot{x} + q(x)] = 0$. Obviously, the locus is symmetric in the x-axis. Since $\dfrac{dv}{dx} = -q(x)/v$, in the first and third quadrants, we have $\frac{\mathrm{d}v}{\mathrm{d}x} < 0$; in the second and fourth quadrants, $\frac{\mathrm{d}v}{\mathrm{d}x} > 0$. For any small value of E, the function $E - V(x)$ has a maximum value E and decreases on both sides of $x = 0$. Since E is very small, as the value of $|x|$ increasing, $E - V(x)$ eventually becomes zero. Hence, there are two points $x = -B$ and $x = A$ at which the function $E - V(x)$ is zero, where A, B are small and positive. These imply that each locus is a simple closed curve, symmetric about the x-axis.

We can show that the integral curves of (2.11) resemble a distorted family of circles $u^2 + v^2 = 2E$ for very small E. To see this, we change of the coordinates by $(x, v) \to (\pm\sqrt{2V(x)}, v)$, according as x is positive or negative. The transformation is of class $\mathcal{C}^1$ and the Jacobian determinant

$$\det \begin{pmatrix} \pm q(x)/\sqrt{2V(x)} & 0 \\ 0 & 1 \end{pmatrix} = \pm q(x)/\sqrt{2V(x)} \neq 0,$$

if $x \neq 0$. As $E \downarrow 0$, we know $|v| = \sqrt{2[E - V(x)]} \leq \sqrt{2E} \to 0$, so the closed integral curves shrink monotonically towards the origin. Hence, the integral curves resemble a distorted family of circles $u^2 + v^2 = 2E$. □

Now we can apply Theorem 2.6.1 to the system $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = x^3 - x$, studied in Example 2.1.2. In this special case, the restoring force $q(x) = x - x^3$ and the origin $(0, 0)$ is a vortex point of this system. Theorem 2.6.1 can also be applied to Example 2.1.3 for the simple pendulum equation $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -k^2 \sin\theta$. A direct application of the theorem implies that the origin is a vortex point. A translation $\theta \to k\pi + \theta$ can be used to show that all critical points at $(k\pi, 0)$ are vortex points, where $k = \pm 2, \pm 4, \ldots$.

2.6.2 Damped Oscillations

For the noninear oscillator (2.10), if $p(x, \dot{x})$ is not identically zero, we have a damped nonlinear oscillator. Let us consider the case $q(x) = xh(x)$ in this subsection. Once again we assume that the restoring force tends to restore equilibrium under static conditions by assuming $h(0) > 0$. The nonlinear oscillator is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -vp(x, v) - xh(x). \end{cases} \tag{2.12}$$

The origin $(0, 0)$ is always a critical point of this plane autonomous system.

Theorem 2.6.2 *If p and h are of class $\mathcal{C}^1$, and if $p(0, 0)$ and $h(0)$ are positive, then the origin is a strictly stable critical point of the damped nonlinear oscillator (2.12).*

Proof The system (2.12) can be rewritten as

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -h(0)x - p(0, 0)v + O(x^2 + v^2).$$

Now the linearized system is

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -h(0)x - p(0, 0)v,$$

whose coefficient matrix is

$$\begin{pmatrix} 0 & 1 \\ -h(0) & -p(0,0) \end{pmatrix}.$$

The eigenvalues of the matrix are

$$\frac{-p(0,0) \pm \sqrt{p^2(0,0) - 4h(0)}}{2}$$

that have negative real parts if $p(0,0) > 0$ and $h(0) > 0$. Hence, by Theorem 2.4.4, the origin is a strictly stable critical point; the solution curves tend to the origin in the vicinity of the origin. □

If $p(0,0) < 0$, the system is said to be negatively damped, and the origin unstable. In fact, if we reverse the time, the differential equation becomes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}(-t)^2} - p\left(x, -\frac{\mathrm{d}x}{\mathrm{d}(-t)}\right)\frac{\mathrm{d}x}{\mathrm{d}(-t)} + h(x)x = 0.$$

Hence, if $p(0,0) < 0$, all solution curves spiral outward near the origin.

It is worth to notice that there exists a weak Liapunov function for the system (2.12). In fact, for

$$V(x,v) = \frac{1}{2}v^2 + \int_0^x s h(s)\mathrm{d}s,$$

it is positive definite for small (x, v) since $h(0) > 0$. Moreover,

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,v)\cdot X(x,v) + \frac{\partial}{\partial v}V(x,v)\cdot Y(x,v) \\ = \;& xh(x)\cdot v + v\cdot(-vp(x,v) - xh(x)) \\ = \;& -v^2 p(x,v) \le 0, \end{aligned}$$

in a neighborhood of the origin. Hence, by Theorem 2.4.3, we know that the origin is stable. However, this Liapunov function is not capable of detecting that the origin is strictly stable.

2.7 MISCELLANEOUS PROBLEMS

1. Construct a strong Liapunov function for the following regular system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3(1+y^2)y \end{cases}$$

and justify your result.

Solution To construct a strong Liapunov function, we first try to make a linear equivalent transform

$$\begin{pmatrix} x \\ y \end{pmatrix} = K \begin{pmatrix} u \\ v \end{pmatrix},$$

where K is the matrix formed by two linearly independent eigenvectors of the coeficient matrix:

$$K = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = K^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = K^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = K^{-1}AK\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Under the same linear transform, the perturbed nonlinear system has the form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = -2u + f(u,v), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -v + g(u,v), \end{cases}$$

where $f(u,v)$ and $g(u,v)$ are polynomials of u and v, with the lowest terms from at least the second order. For the later system, by the discussion in the proof of Theorem 2.4.4, the function $u^2 + v^2$ is a strong Liapunov function. Since the inverse transform gives $u = -x - y$ and $v = 2x + y$, thus, we have a strong Liapunov function of the original nonlinear system:

$$V(x,y) = (-x-y)^2 + (2x+y)^2 = 5x^2 + 6xy + 2y^2.$$

To see that it is indeed a strong Liapunov function, we only need to check that

$$\begin{aligned} & \frac{\partial}{\partial x}V(x,y)\cdot X(x,y) + \frac{\partial}{\partial y}V(x,y)\cdot Y(x,y) \\ = \;& (10x+6y)y + (6x+4y)\left(-2x - 3(1+y^2)y\right) \\ = \;& -2\left(6x^2 + 8xy + 3y^2 + 9xy^3 + 6y^4\right) \\ \le \;& -\left(6x^2 + 8xy + 3y^2\right) < 0. \end{aligned}$$

Here the inequalities hold since $9xy^3 + 6y^4$ is of higher order than 2 and $6x^2 + 8xy + 3y^2$ is positive definite. A similar argument can be made as in the proof of Theorem 2.4.4. □