As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for *a particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x-y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x-y, t)$ and so is

$$v(x,t)=\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x,t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\,x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x,t) = u(\sqrt{a}\,x, at)$. Then $v_t = [\partial(at)/\partial t]u_t = au_t$ and $v_x = [\partial(\sqrt{a}\,x)/\partial x]u_x = \sqrt{a}\,u_x$ and $v_{xx} = \sqrt{a}\cdot\sqrt{a}\,u_{xx} = a\,u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$Q(x,0)=1 \quad \text{for } x>0 \qquad Q(x,0)=0 \quad \text{for } x<0. \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x,t)=g(p) \quad \text{where } p=\frac{x}{\sqrt{4kt}} \tag{4}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that equation (1) doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, (3) doesn't change at all under the dilation. So $Q(x, t)$, which is defined by conditions (1) and (3), ought not see the dilation either. How could that happen? In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus let $p = x/\sqrt{4kt}$ and look for Q which satisfies (1) and (3) and has the form (4).

Step 2 Using (4), we convert (1) into an ODE for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p)$$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right].$$

Thus

$$g'' + 2pg' = 0.$$

This ODE is easily solved using the integrating factor $\exp \int 2p\,dp = \exp(p^2)$. We get $g'(p) = c_1 \exp(-p^2)$ and

$$Q(x, t) = g(p) = c_1 \int e^{-p^2}dp + c_2.$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(x, t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}dp + c_2.$$

This formula is valid only for $t > 0$. Now use (3), expressed as a limit as follows.

$$\text{If } x > 0, \quad 1 = \lim_{t \searrow 0} Q = c_1 \int_0^{+\infty} e^{-p^2}dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2.$$

$$\text{If } x < 0, \quad 0 = \lim_{t \searrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2}dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2.$$

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, expressed as a limit as

$2 = c_1 \frac{\sqrt{\pi}}{2} + c_2$.

$2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2$.

See Exercise 6. Here $\lim_{t \searrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp \tag{5}$$

for $t > 0$. Notice that it does indeed satisfy (1), (3), and (4).

Step 4 Having found Q, we now *define* $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By property (b), S is also a solution of (1). Given any function ϕ, we also *define*

diffusion kernel

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad \text{for } t > 0. \tag{6}$$

By property (d), u is another solution of (1). *We claim that u is the unique solution of* (1), (2). To verify the validity of (2), we write

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\,dy \\ &= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=+\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy \\ &= \int_{-\infty}^{x} \phi'(y)\,dy = \phi\Big|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition (2). We conclude that (6) is our solution formula, where

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad \text{for } t > 0. \tag{7}$$

That is,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\,dy. \tag{8}$$