Math2131 Final, Autumn 2015

(1) Let

$$A = \begin{pmatrix} 0 & 1 & 2 & -3 \\ 1 & 2 & -3 & 0 \\ 2 & -3 & 0 & 1 \\ -3 & 0 & 1 & 2 \end{pmatrix}.$$

1. Find the matrices for the orthogonal projections to Ran$A$ and Ker$A$.
2. Find the matrix of the linear operator that has Ran$A$ and Ker$A$ as eigenspaces with eigenvalues 1 and 2.

(2) Let

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}.$$

1. Find an orthonormal basis of eigenvectors for A.
2. Find the orthogonal projections of $\vec{b}$ onto the eigenspaces of A.
3. Find the quadratic form $q(x_1, x_2, x_3)$ corresponding to A and use an orthogonal change of variable (write explicit formula between two sets of variables) to eliminte cross terms in q.

(3)

1. Prove that the matrix of orthogonal projection to a line (1-dimensional subspace) in $\mathbb{R}^n$ is precisely $\vec{v}\vec{v}^T$ (product of $n \times 1$ matrix and $1 \times n$ matrix) for a unit length vector $\vec{v}$.
2. Prove that the sum of two positive semi-definite matrices is positive semi-definite.
3. Use the first and second parts to prove that a matrix is positive semi-definite if and only if it is a sum of matrices of the form $\vec{v}\vec{v}^T$ ($\vec{v}$ may not have unit length).

(4) Let L be the linear operator on P_n, such that odd polynomials are eigenvectors with eigenvalue 1 and even polynomials are eigenvectors with eigenvalue -1.

1. What is the characteristic polynomial of L?
2. Find second order polynomial satisfying $L^2 + bL + cI = O$.
3. Find the matrix of L with respect to the basis $1, (t-1), (t-1)^2, \ldots, (t-1)^n$.

Answer to Math 2131 Final, Autumn 2015

(1.1) We have Ker$A = \mathbb{R}\vec{v}$, $\vec{v} = (1\ 1\ 1\ 1)^T$. The orthogonal projection to Ker$A$ is

$$P_1\vec{x} = \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{x_1+x_2+x_3+x_4}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\vec{x}.$$

The orthogonal projection to Ran$A$ = RanA^T = (Ker$A$)$^\perp$ (A is symmetric) is

$$P_2\vec{x} = \vec{x} - P_1\vec{x} = \frac{1}{4}\begin{pmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{pmatrix}\vec{x}.$$

(1.2) The linear operator is $P_2 + 2P_1$ and has matrix

$$= \frac{1}{4}\begin{pmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{pmatrix} + 2\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}5&1&1&1\\1&5&1&1\\1&1&5&1\\1&1&1&5\end{pmatrix}.$$

(2.1) We have

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{pmatrix}\lambda&-2&1\\-2&\lambda-3&2\\1&2&\lambda\end{pmatrix} \overset{C_2-2C_1}{=} \det\begin{pmatrix}\lambda&-2\lambda-2&1\\-2&\lambda+1&2\\1&0&\lambda\end{pmatrix}\\ &\overset{R_1+2R_2}{=} \det\begin{pmatrix}\lambda-4&0&5\\-2&\lambda+1&2\\1&0&\lambda\end{pmatrix} = (\lambda+1)\det\begin{pmatrix}\lambda-4&5\\1&\lambda\end{pmatrix}\\ &= (\lambda+1)(\lambda^2-4\lambda-5) = (\lambda+1)^2(\lambda-5).\end{aligned}$$

For the eigenvalue 5, we have

$$5I - A = \begin{pmatrix}5&-2&1\\-2&2&2\\1&2&5\end{pmatrix},$$

and the eigenspace Ker$(5I - A)$ has basis $\vec{v}_1 = (-1\ -2\ 1)^T$. For the eigenvalue -1, we have

$$-I - A = \begin{pmatrix}-1&-2&1\\-2&-4&2\\1&2&-1\end{pmatrix},$$

and the eigenspace Ker$(-I - A)$ has basis $\vec{v}_2 = (1\ 0\ 1)^T$, $\vec{v}_3 = (-2\ 1\ 0)^T$.

Because A is symmetric, the two eigenspaces are orthogonal. We may use Gram-Schmidt process to further make $\vec{v}_2, \vec{v}_3$ orthogonal

$$\vec{v}_2' = \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\vec{v}_3' = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Then after dividing the lengths, we get an orthonormal basis of eigenvectors

$$\vec{u}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

This is the same as

$$A = UDU^{-1} = UDU^T, \quad U = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(2.2) Since $\mathrm{Ker}(5I - A) = \mathbb{R}\vec{v}_1$, the orthogonal projection of $\vec{b}$ on the first eigenspace is

$$\mathrm{proj}_{\mathrm{Ker}(5I-A)}\vec{b} = \frac{\vec{b} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{0 \cdot (-1) + (-1) \cdot (-2) + 4 \cdot 1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Since the second eigenspace $\mathrm{Ker}(-I - A) = (\mathrm{Ker}(5I - A))^{\perp}$ is the orthogonal complement of the first eigenspace, we get

$$\mathrm{proj}_{\mathrm{Ker}(-I-A)}\vec{b} = \vec{b} - \mathrm{proj}_{\mathrm{Ker}(5I-A)}\vec{b} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

(2.3) The quadratic form corresponding to A is

$$q(x_1, x_2, x_3) = 3x_2^2 + 4x_1x_2 - 2x_1x_3 - 4x_2x_3.$$

By $q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{x}^T UDU^T\vec{x} = (U^T\vec{x})^T D(U^T\vec{x})$, we find that, if $\vec{y} = U^T\vec{x}$, in other words,

$$y_1 = \frac{1}{\sqrt{6}}(-x_1 - 2x_2 + x_3),$$
$$y_2 = \frac{1}{\sqrt{2}}(x_1 + x_3),$$
$$y_3 = \frac{1}{\sqrt{3}}(-x_1 + x_2 + x_3),$$

then

$$q(\vec{x}) = \vec{y}^T D\vec{y} = 5y_1^2 - y_2^2 - y_3^2.$$

Note: By $U^{-1} = U^T$, $\vec{y} = U^T\vec{x}$ is the same as $\vec{x} = U\vec{y}$. So the change of variable is also given by the formula

$$\begin{aligned}
x_1 &= -\frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 - \frac{1}{\sqrt{3}}y_3, \\
x_2 &= -\frac{2}{\sqrt{6}}y_1 + \frac{1}{\sqrt{3}}y_3, \\
x_3 &= \frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3.
\end{aligned}$$

(3.1) Let the 1-dimensional line be the span of unit length vector $\vec{v}$. Then

$$\text{proj}_{\mathbb{R}\vec{v}}\vec{x} = (\vec{x}\cdot\vec{v})\vec{v} = \vec{v}(\vec{v}\cdot\vec{x}) = \vec{v}(\vec{v}^T\vec{x}) = (\vec{v}\vec{v}^T)\vec{x}.$$

Here $(\vec{x}\cdot\vec{v})\vec{v}$ is the vector $\vec{v}$ multiplied by a scalar $\vec{x}\cdot\vec{v}$, and $\vec{v}(\vec{v}\cdot\vec{x})$ is $n\times 1$ matrix $\vec{v}$ multiplied to 1×1 matrix $\vec{v}\cdot\vec{x}$. The last $(\vec{v}\vec{v}^T)\vec{x}$ is a matrix multiplication, and the matrix of $\text{proj}_{\mathbb{R}\vec{v}}$ is $\vec{v}\vec{v}^T$.

(3.2) A matrix A is positive semi-definite if and only if $A\vec{x}\cdot\vec{x} \geq 0$ for all $\vec{x}$. Then for positive semi-definite A and B, we have

$$(A+B)\vec{x}\cdot\vec{x} = (A\vec{x}+B\vec{x})\cdot\vec{x} = A\vec{x}\cdot\vec{x} + B\vec{x}\cdot\vec{x} \geq 0.$$

Therefore $A+B$ is positive semi-definite.

(3.3) If A is positive semi-definite, then we have orthonormal basis $\vec{v}_1,\dots,\vec{v}_n$ of eigenvectors with non-negative eigenvalues $d_1,\dots,d_n$. We have

$$A(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = d_1x_1\vec{v}_1 + \cdots + d_nx_n\vec{v}_n.$$

Using the first part, this means that

$$A = d_1\text{proj}_{\mathbb{R}\vec{v}_1} + \cdots + d_n\text{proj}_{\mathbb{R}\vec{v}_n} = d_1\vec{v}_1\vec{v}_1^T + \cdots + d_n\vec{v}_n\vec{v}_n^T = \vec{w}_1\vec{w}_1^T + \cdots + \vec{w}_n\vec{w}_n^T, \quad \vec{w}_i = \sqrt{d_i}\vec{v}_i.$$

Conversely, by second part, we only need to show that the matrix $\vec{v}\vec{v}^T$ is positive semi-definite. This follows from

$$(\vec{v}\vec{v}^T\vec{x})\cdot\vec{x} = (\vec{v}\vec{v}^T\vec{x})^T\vec{x} = \vec{x}^T\vec{v}\vec{v}^T\vec{x} = (\vec{v}\cdot\vec{x})^2 \geq 0.$$

A more conceptual proof of the converse is that any orthogonal projection $P = \text{proj}_H$ is positive semi-definite. Any vector $\vec{x}$ can be written $\vec{x} = \vec{h} + \vec{w}$, $\vec{h}\in H$ and $\vec{w}\in H^\perp$. Then $P\vec{x} = \vec{h}$, and

$$\langle P\vec{x},\vec{x}\rangle = \langle \vec{h},\vec{h}+\vec{w}\rangle = \langle \vec{h},\vec{h}\rangle = \|\vec{h}\|^2 \geq 0.$$

The second equality uses $\vec{h}\perp\vec{w}$. Note that if $\vec{v} = a\vec{u}$, where $\vec{u}$ has unit length, then $\vec{v}\vec{v}^T = a^2\vec{u}\vec{u}^T = a^2\text{proj}_{\mathbb{R}\vec{u}}$. As non-negative constant multiple of a projection, $\vec{v}\vec{v}^T$ is therefore positive semi-definite.

(4.1) The eigenspaces are

$$\operatorname{Ker}(I-L)=\{a_1t+a_3t^3+a_5t^5+\cdots\},$$
$$\operatorname{Ker}(-I-L)=\{a_0+a_2t^2+a_4t^4+\cdots\}.$$

We have $P_n=\operatorname{Ker}(I-L)\oplus\operatorname{Ker}(-I-L)$, and L is diagonalisable. The dimensions of the eigenspaces are

$$d_+=\dim\operatorname{Ker}(I-L)=\begin{cases}\frac{n+1}{2}, & n \text{ odd}\\ \frac{n}{2}, & n \text{ even}\end{cases}\qquad d_-=\dim\operatorname{Ker}(-I-L)=\begin{cases}\frac{n+1}{2}, & n \text{ odd}\\ \frac{n+2}{2}, & n \text{ even}\end{cases}$$

The characteristic polynomial of L is

$$(t-1)^{d_+}(t+1)^{d_-}=\begin{cases}(t-1)^{\frac{n+1}{2}}(t+1)^{\frac{n+1}{2}}, & n \text{ odd}\\ (t-1)^{\frac{n}{2}}(t+1)^{\frac{n+2}{2}}, & n \text{ even}\end{cases}$$

(4.2) We know L^2 is identity on both eigenspaces. Therefore L^2 is identity on the whole space. We get $L^2-I=O$.

(4.3) Using binomial expansion, we have

$$\begin{aligned}L((t-1)^k)&=L(t^k)-\binom{k}{k-1}L(t^{k-1})+\binom{k}{k-2}L(t^{k-2})+\\&\quad\cdots+(-1)^{k-1}\binom{k}{1}L(t)+(-1)^k\binom{k}{0}L(1)\\&=(-1)^k\left[t^k+\binom{k}{k-1}t^{k-1}+\binom{k}{k-2}t^{k-2}+\cdots+\binom{k}{1}t+\binom{k}{0}\right]\\&=(-1)^k(t+1)^k=(-1)^k((t-1)+2)^k\\&=(-1)^k\left[(t-1)^k+\binom{k}{k-1}2(t-1)^{k-1}+\binom{k}{k-2}2^2(t-1)^{k-2}+\right.\\&\quad\left.\cdots+\binom{k}{1}2^{k-1}(t-1)+\binom{k}{0}2^k\right].\end{aligned}$$

The coefficients form the column of the matrix of L with respect to the basis $1,(t-1),(t-1)^2,\dots,(t-1)^n$

$$[L]=\begin{pmatrix}\binom{0}{0} & -\binom{1}{1} & \binom{2}{2} & -\binom{3}{3} & \cdots & (-1)^n\binom{n}{n}\\ 0 & -\binom{1}{0}2 & \binom{2}{1}2 & -\binom{3}{2}2 & \cdots & (-1)^n\binom{n}{n-1}2\\ 0 & 0 & \binom{2}{0}2^2 & -\binom{3}{1}2^2 & \cdots & (-1)^n\binom{n}{n-2}2^2\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & (-1)^n\binom{n}{0}2^n\end{pmatrix}.$$

A conceptual way of seeing $L((t-1)^k)=(-1)^k(t+1)^k$ is that the eigenvector assumption on L means $L(t^k)=(-t)^k$. Therefore L is an isomorphism of the polynomial algebra $\mathbb{R}[t]$ by sending t to $-t$. This means that $L(p(t)) = p(-t)$, and in particular, $L((t-1)^k) = (-t-t)^k = (-1)^k(t+1)^k$.

Math 2131 Final, Autumn 2017

(1) Express the symmetric matrix

$$\begin{pmatrix} 3 & 2 & -2 \\ 2 & 0 & 4 \\ -2 & 4 & 0 \end{pmatrix}$$

as UDU^{-1}, where D is a diagonal matrix, and U is an orthogonal matrix.
(2) Consider the inner product on $\mathbb{R}^2$ (see Example 4.1.2)

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5y_1y_2 = \vec{x} \cdot Q\vec{y}, \quad Q = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

1. Express $\langle \vec{x}, \vec{x} \rangle$ as $(a_{11}x_1 + a_{12}x_2)^2 + (a_{21}x_1 + a_{22}x_2)^2$.
2. Find the matrix P of a linear operator that gives isometry $\mathbb{R}^2_{\langle\cdot,\cdot\rangle} \to \mathbb{R}^2_{\text{dot}}$.
3. Show that $P^TP = Q$.
4. Find an orthonormal basis of $\mathbb{R}^2$ with respect to the new inner product.
5. Suppose the matrix of a linear operator L on $\mathbb{R}^2$ is A. What is the matrix of the adjoint L^* with respect to the new inner product on $\mathbb{R}^2$?

(3) Let P be the orthogonal projection of $\mathbb{R}^3$ to the subspace H given by $x + y + z = 0$. Let L be a linear operator on $\mathbb{R}^3$ satisfying

1. L is self-adjoint.
2. $LP = PL$.
3. L has three eigenvalues $0, 1, 2$.
4. $L(1, -1, 0) = \vec{0}$.

What can be the matrix of L? (It is enough to give PDP^{-1}. No need to calculate the matrix itself.)
(4) Suppose $L = L_1 \oplus L_2$ with respect to $V = V_1 \oplus V_2$.

1. Prove that L is diagonalisable if and only if L_1 and L_2 are diagonalisable.
2. If V is an inner product space and $V_1 \perp V_2$, prove that L is orthogonally diagonalisable if and only if L_1 and L_2 are orthogonally diagonalisable.

Answer to Math 2131 Final, Autumn 2017

(1) We have

$$\det(tI-A)=\det\begin{pmatrix}t-3&-2&2\\-2&t&-4\\2&-4&t\end{pmatrix}=\det\begin{pmatrix}t-3&-2&2\\-2&t&-4\\0&t-4&t-4\end{pmatrix}$$
$$=\det\begin{pmatrix}t-3&-4&2\\-2&t+4&-4\\0&0&t-4\end{pmatrix}=[(t-3)(t+4)-8](t-4)=(t-4)^2(t+5).$$

Then (try to construct orthogonal solutions, instead of finding solution and then apply Gram-Schmidt)

$$A-4I=\begin{pmatrix}-1&2&-2\\2&-4&4\\-2&4&-4\end{pmatrix},\quad \text{Ker}(A-4I)=\mathbb{R}(0,1,1)\perp\mathbb{R}(4,1,-1),$$

and (this can be more easily obtained by orthogonal to $\text{Ker}(A-4I)$)

$$A+5I=\begin{pmatrix}8&2&-2\\2&5&4\\-2&4&5\end{pmatrix},\quad \text{Ker}(A+5I)=\mathbb{R}(1,-2,2),$$

After dividing the length, we get

$$A=UDU^{-1},\quad D=\begin{pmatrix}4&0&0\\0&4&0\\0&0&-5\end{pmatrix},\quad U=\begin{pmatrix}0&\frac{4}{\sqrt{18}}&\frac{1}{3}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{18}}&-\frac{2}{3}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{18}}&\frac{2}{3}\end{pmatrix}.$$

(2.1) $\langle\vec{x},\vec{x}\rangle=x_1^2+4x_1x_2+5x_2^2=(x_1+2x_2)^2+x_2^2$.

(2.2) By (2.1), we see that $P=\begin{pmatrix}1&2\\0&1\end{pmatrix}$ satisfies $\langle\vec{x},\vec{x}\rangle=P\vec{x}\cdot P\vec{x}$. This implies $\langle\vec{x},\vec{y}\rangle=P\vec{x}\cdot P\vec{y}$, and can be interpreted as $P\colon\mathbb{R}^2_{\langle\cdot,\cdot\rangle}\to\mathbb{R}^2_{\text{dot}}$ being an isometry.

(2.3) We have

$$\vec{x}\cdot Q\vec{y}=\langle\vec{x},\vec{y}\rangle=P\vec{x}\cdot P\vec{y}=\vec{x}\cdot P^TP\vec{y}.$$

The first equality is the definition of the new inner product, the second is from (2.2), and the third is that the usual adjoint with respect to the dot product is transpose. Compare the two sides, we get $Q=P^TP$.

(2.4) If we take the standard basis of $\mathbb{R}^2$, then

$$P^{-1}\vec{e}_1=\begin{pmatrix}1&-2\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad P^{-1}\vec{e}_2=\begin{pmatrix}1&-2\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-2\\1\end{pmatrix}$$

is an orthonormal basis with respect to the new inner product.

(2.5) Let B be the adjoint of A with respect to the new inner product on $\mathbb{R}^2$. This means $\langle A\vec{x},\vec{y}\rangle=\langle\vec{x},B\vec{y}\rangle$, or $A\vec{x}\cdot Q\vec{y}=\vec{x}\cdot QB\vec{y}$. By the usual adjoint with respect to the dot product being transpose, this means $\vec{x} \cdot A^T Q\vec{y} = \vec{x} \cdot QB\vec{y}$. Therefore $A^T Q = QB$, or $B = Q^{-1}A^T Q$.

(3) By the geometric meaning of P, the eigenspaces of P are

$$\text{Ker}P = \mathbb{R}(1,1,1), \quad \text{Ker}(P-I) = \mathbb{R}(1,-1,0) \perp \mathbb{R}\vec{v},$$

where $(1,1,1)$ is the coefficient of $x+y+z=0$, $(1,-1,0)$ is the vector in item (4), and $\{(1,1,1),(1,-1,0),\vec{v}\}$ form an orthogonal basis. Since $LP = PL$, the eigenspaces are invariant subspaces of L. This means that $(1,1,1)$ is an eigenvector of L. By item (4), also know $(1,-1,0)$ is an eigenvector of L of eigenvalue 0. Since L is self-adjoint, it has an orthogonal basis of eigenvectors. This implies that $\vec{v}$ is also an eigenvector of L.

So we know $\{(1,1,1),(1,-1,0),\vec{v}\}$ is an orthogonal basis of eigenvectors of L, with the second vector having eigenvalue 0. By item (3), we know $(1,1,1),\vec{v}$ have eigenvalues $1,2$.

It is easy to get $\vec{v} = (1,1,-2)$ (see Example 4.2.10). If $L(1,1,1) = (1,1,1)$ and $L(\vec{v}) = 2\vec{v}$, then (second line not needed for exam)

$$\begin{aligned}[L] &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}.\end{aligned}$$

If $L(1,1,1) = 2(1,1,1)$ and $L(\vec{v}) = \vec{v}$, then

$$[L] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 8 \end{pmatrix}.$$

(4) For any vector $\vec{v} = \vec{v}_1 + \vec{v}_2 \in V$, with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$, we have

$$L(\vec{v}) = L(\vec{v}_1) + L(\vec{v}_2), \quad \lambda\vec{v} = \lambda\vec{v}_1 + \lambda\vec{v}_2.$$

By the direct sum $V_1 \oplus V_2$, we find $L(\vec{v}) = \lambda\vec{v}$ if and only if $L(\vec{v}_1) = \lambda\vec{v}_1$ and $L(\vec{v}_2) = \lambda\vec{v}_2$. This proves that

$$\text{Ker}(L-\lambda I) = \text{Ker}(L_1 - \lambda I) \oplus \text{Ker}(L_2 - \lambda I).$$

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be all the (distinct) eigenvalues of L. Then by the equality above, the possible eigenvalues of L_1 and L_2 are also among these. Moreover, we have

$$\begin{aligned}\text{Ker}(L-\lambda_1 I) \oplus \cdots \oplus \text{Ker}(L-\lambda_k I) &= (\text{Ker}(L_1-\lambda_1 I) \oplus \cdots \oplus \text{Ker}(L_1-\lambda_k I)) \\ &\quad \oplus (\text{Ker}(L_2-\lambda_1 I) \oplus \cdots \oplus \text{Ker}(L_2-\lambda_k I)).\end{aligned}$$

The left is a subspace of V, and the right is a direct sum of a subspace of V_1 and a subspace of V_2. By $V = V_1 \oplus V_2$, we then get

$$L \text{ is diagonalisable}: V = \text{Ker}(L-\lambda_1 I) \oplus \cdots \oplus \text{Ker}(L-\lambda_k I)$$

if and only if

$$L_1 \text{ is diagonalisable}\colon V_1 = \mathrm{Ker}(L_1 - \lambda_1 I) \oplus \cdots \oplus \mathrm{Ker}(L_1 - \lambda_k I)$$

and

$$L_2 \text{ is diagonalisable}\colon V_2 = \mathrm{Ker}(L_2 - \lambda_1 I) \oplus \cdots \oplus \mathrm{Ker}(L_2 - \lambda_k I).$$

If $V_1 \perp V_2$, then we have

$$\mathrm{Ker}(L - \lambda I) = \mathrm{Ker}(L_1 - \lambda I) \perp \mathrm{Ker}(L_2 - \lambda I).$$

Moreover, the orthogonal diagonalisability means the decomposition of vector space into orthogonal sum of eigenspaces. Therefore we may change all $\oplus$ above into $\perp$, and conclude that L is orthogonally diagonalisable if and only if L_1 and L_2 are orthogonally diagonalisable.