SECTION 1.1

Exercise 1.1

$$\begin{aligned}(a+b)(\vec{x}+\vec{y}) &= (a+b)\vec{x}+(a+b)\vec{y} && \text{(Axiom 7)}\\ &= (a\vec{x}+b\vec{x})+(a\vec{y}+b\vec{y}) && \text{(Axiom 8)}\\ &= a\vec{x}+b\vec{y}+b\vec{x}+a\vec{y}. && \text{(Axioms 1 and 2)}\end{aligned}$$

Exercise 1.2

1. $(x_1,x_2)+(y_1,y_2)=(x_1+y_2,x_2+y_1)$ and $(y_1,y_2)+(x_1,x_2)=(y_1+x_2,y_2+x_1)$. Axiom 1 not satisfied.

2. $((x_1,x_2)+(y_1,y_2))+(z_1,z_2)=(x_1+y_2,x_2+y_1)+(z_1,z_2)=(x_1+y_2+z_1,x_2+y_1+z_1)$ and $(x_1,x_2)+((y_1,y_2)+(z_1,z_2))=(x_1,x_2)+(y_1+z_2,y_2+z_1)=(x_1+y_2+z_1,x_2+y_1+z_2)$. Axiom 2 not satisfied.

3. $(x_1,x_2)+(0,0)=(x_1,x_2)=(0,0)+(x_1,x_2)$. Axiom 3 satisfied for $\vec{0}=(0,0)$.

4. $1(x_1,x_2)+(-x_2,x_1)=(0,0)=(-x_2,x_1)+(x_1,x_2)$. Axiom 4 satisfied.

5. $1(x_1,x_2)=(1x_1,1x_2)=(x_1,x_2)$. Axiom 5 satisfied.

6. $a(b(x_1,x_2))=a(bx_1,bx_2)=(a(bx_1),a(bx_2))=((ab)x_1,(ab)x_2)=(ab)(x_1,x_2)$. Axiom 6 satisfied.

7. $(a+b)(x_1,x_2)=((a+b)x_1,(a+b)x_2)=(ax_1+bx_1,ax_2+bx_2)$ and $a(x_1,x_2)+b(x_1,x_2)=(ax_1,ax_2)+(bx_1,bx_2)=(ax_1+bx_2,ax_2+bx_1)$. Axiom 7 not satisfied.

8. $a(x_1,x_2)+a(y_1,y_2)=(ax_1,ax_2)+(ay_1,ay_2)=(ax_1+ay_2,ax_2+ay_1)=a(x_1+y_2,x_2+y_1)=a((x_1,x_2)+(y_1,y_2))$. Axiom 8 satisfied.

Exercise 1.3

1. $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,0)=(y_1+x_1,0)=(y_1,y_2)+(x_1,x_2)$. Axiom 1 satisfied.

2. $((x_1,x_2)+(y_1,y_2))+(z_1,z_2)=(x_1+y_1,0)+(z_1,z_2)=(x_1+y_1+z_1,0)$ and $(x_1,x_2)+((y_1,y_2)+(z_1,z_2))=(x_1,x_2)+(y_1+z_1,0)=(x_1+y_1+z_1,0)$. Axiom 2 satisfied.

3. $(x_1,x_2)+\text{any vector}=(x,0)\neq(x_1,x_2)$. Axiom 3 not satisfied.

4. By no $\vec{0}$, Axiom 4 does not make sense.

5. $1(x_1,x_2)=(1x_1,1x_2)=(x_1,x_2)$. Axiom 5 satisfied.

6. $a(b(x_1,x_2))=a(bx_1,0)=(a(bx_1),0)=((ab)x_1,0)=(ab)(x_1,x_2)$. Axiom 6 satisfied.

7. $(a+b)(x_1,x_2) = ((a+b)x_1,0) = (ax_1+bx_1,0) = (ax_1,0)+(bx_2,0) = a(x_1,x_2)+b(x_1,x_2)$. Axiom 7 satisfied.

8. $a(x_1,x_2) + a(y_1,y_2) = (ax_1,0) + (ay_1,0) = (ax_1 + ay_1, 0) = a(x_1 + y_1, 0) = a((x_1,x_2) + (y_1,y_2))$. Axiom 8 satisfied.

Exercise 1.4

By Axiom 1, we know $(x_1,x_2) + (y_1,y_2) = (x_1 + ky_1, x_2 + ly_2)$ and $(y_1,y_2) + (x_1,x_2) = (y_1 + kx_1, y_2 + lx_2)$ are equal. Therefore we have $x_1 + ky_1 = y_1 + kx_1$ and $x_2 + ly_2 = y_2 + lx_2$ for all x_1, x_2, y_1, y_2. This implies $k = l = 1$.

Conversely, if $k = l = 1$, then by Exercises 1.1.2, we have the Euclidean space, satisfying all axioms.

Exercise 1.5

$$\begin{aligned}
(x_n) + (y_n) &= (x_n + y_n) = (y_n + x_n) = (y_n) + (x_n);\\
((x_n) + (y_n)) + (z_n) &= ((x_n + y_n) + z_n) = (x_n + (y_n + z_n)) = (x_n) + ((y_n) + (z_n));\\
(x_n) + (0) &= (x_n + 0) = (x_n);\\
(x_n) + (-x_n) &= (x_n + (-x_n)) = (0);\\
1(x_n) &= (1x_n) = (x_n);\\
a(b(x_n)) &= a(bx_n) = (a(bx_n)) = ((ab)x_n) = (ab)(x_n);\\
(a+b)(x_n) &= ((a+b)x_n) = (ax_n + bx_n) = (ax_n) + (bx_n) = a(x_n) + b(x_n);\\
a((x_n) + (y_n)) &= a(x_n + y_n) = (a(x_n + y_n)) = (ax_n + ay_n) = (ax_n) + (ay_n) = a(x_n) + a(y_n).
\end{aligned}$$

Exercise 1.6

If $f(t)$ and $g(t)$ are smooth, then $f(t) + g(t)$ and $af(t)$ are smooth. If $f(t)$ and $g(t)$ are even, then $f(t) + g(t)$ is even by $f(-t) + g(-t) = f(t) + g(t)$, and $af(t)$ is even by $(af)(-t) = a(f(-t)) = a(f(t)) = (af)(t)$. Moreover, the zero function is even, and $-f(t)$ is also even. Then the eight axioms are verified juts like for all functions.

Exercise 1.7

Let us use

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix},$$

as example. The following shows $(A + B)^T = A^T + B^T$.

$$(A+B)^T = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \\ a_{13}+b_{13} & a_{23}+b_{23} \end{pmatrix},$$

$$A^T + B^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \\ a_{13}+b_{13} & a_{23}+b_{23} \end{pmatrix}.$$

The following verifies $(cA)^T = cA^T$

$$(cA)^T = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}^T = \begin{pmatrix} ca_{11} & ca_{21} \\ ca_{12} & ca_{22} \\ ca_{13} & ca_{23} \end{pmatrix} = c \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} = cA^T.$$

The following verifies $(A^T)^T = A$

$$(A^T)^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = A.$$

Exercise 1.9

Suppose $\vec{v}_1$ and $\vec{v}_2$ are two negative vectors. Then we have

$$\begin{aligned} \vec{u} + \vec{v}_1 &= \vec{0} = \vec{v}_1 + \vec{u}, \\ \vec{u} + \vec{v}_2 &= \vec{0} = \vec{v}_2 + \vec{u}. \end{aligned}$$

This implies

$$\begin{aligned} \vec{v}_1 &= \vec{v}_1 + \vec{0} = \vec{v}_1 + (\vec{u} + \vec{v}_2) \\ &= (\vec{v}_1 + \vec{u}) + \vec{v}_2 = \vec{0} + \vec{v}_2 = \vec{v}_2. \end{aligned}$$

We have

$$\vec{u} + (-1)\vec{u} = 1\vec{u} + (-1)\vec{u} = (1 + (-1))\vec{u} = 0\vec{u} = \vec{0}.$$

The last equality is by Proposition 1.1.4. We have $(-1)\vec{u} + \vec{u} = \vec{0}$ by the similar argument. This means $(-1)\vec{u}$ is the negative vector in Axiom 4.

Finally, the equality $\vec{u} + \vec{v} = \vec{0} = \vec{v} + \vec{u}$ in Axiom 4 is symmetric in $\vec{u}$ and $\vec{v}$. Therefore the equality is the definition of $\vec{v} = -\vec{u}$, and is also the definition of $\vec{u} = -\vec{v}$. Then we have $-(-\vec{u}) = -\vec{v} = \vec{u}$.

Exercise 1.10

By Exercise 1.9, we have

$$a\vec{v} = b\vec{v} \iff \vec{0} = a\vec{v} - b\vec{v} = a\vec{v} + (-1)(b\vec{v}) = a\vec{v} + ((-1)b)\vec{v} = a\vec{v} + (-b)\vec{v} = (a-b)\vec{v}.$$

Then by Proposition 1.1.4, this is equivalent to $a - b = 0$ or $\vec{v} = 0$.

Exercise 1.11

Let $\vec{w}$ be the negative of $\vec{u}$. Then

$$\begin{aligned} \vec{u} + \vec{v}_1 = \vec{u} + \vec{v}_2 &\implies \vec{w} + (\vec{u} + \vec{v}_1) = \vec{w} + (\vec{u} + \vec{v}_2) \\ &\implies (\vec{w} + \vec{u}) + \vec{v}_1 = (\vec{w} + \vec{u}) + \vec{v}_2 && \text{(Axiom 2)} \\ &\implies \vec{0} + \vec{v}_1 = \vec{0} + \vec{v}_2 && \text{(Axiom 4)} \\ &\implies \vec{v}_1 = \vec{v}_2. && \text{(Axiom 3)} \end{aligned}$$

Exercise 1.12

Using $-\vec{u} = (-1)\vec{u}$, we get

$$-(\vec{u} - \vec{v}) = (-1)(\vec{u} + (-1)\vec{v}) = (-1)\vec{u} + ((-1)(-1))\vec{v} = -\vec{u} + \vec{v}.$$

We also get

$$-(\vec{u} + \vec{v}) = (-1)(\vec{u} + \vec{v}) = (-1)\vec{u} + (-1)\vec{v} = -\vec{u} + (-\vec{v}) = -\vec{u} - \vec{v}.$$

SECTION 1.2

Exercise 1.13

If $\vec{u} = c\vec{v}$, then $a\vec{u} + b\vec{v} = (a + bc)\vec{u}$. Therefore the linear combinations is the line $\mathbb{R}\vec{v}$ in the direction of $\vec{v}$.

Exercise 1.14 problem changed to the following

Find the condition on a, such that the last vector can be expressed as a linear combination of the previous ones.

1. $(1, 2, 3), (4, 5, 6), (7, a, 9), (10, 11, 12)$.
2. $(1, 2, 3), (7, a, 9), (10, 11, 12)$.
3. $1 + 2t + 3t^2, 7 + at + 9t^2, 10 + 11t + 12t^2$.
4. $t^2 + 2t + 3, 7t^2 + at + 9, 10t^2 + 11t + 12$.
5. $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & a \\ a & 9 \end{pmatrix}, \begin{pmatrix} 10 & 11 \\ 11 & 12 \end{pmatrix}$.
6. $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 7 & a \\ 9 & 9 \end{pmatrix}, \begin{pmatrix} 10 & 11 \\ 12 & 12 \end{pmatrix}$.

(1) By the row operation

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & a & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & -3 & a-14 & -8 \\ 0 & -6 & -12 & -18 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & a-14 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & 1 \end{pmatrix},$$

The last column is not pivot if and only if $a \neq 8$. This is the condition for the last vector to be a linear combination of the first three.

(2) By the row operaiton in (1), we have

$$\begin{pmatrix} 1 & 7 & 10 \\ 2 & a & 11 \\ 3 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 7 & 10 \\ 0 & 2 & 3 \\ 0 & a-8 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 7 & 10 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{1}{2}(26-3a) \end{pmatrix}.$$

The last column is not pivot if and only if $a \neq \frac{26}{3}$. This is the condition for the last vector to be a linear combination of the first two.

(3) By $a_0 + a_1 t + a_2 t^2 \leftrightarrow (a_0, a_1, a_2)$, the problem is translated into (2). The condition is $a \neq \frac{26}{3}$.

(4) By $a_0 t^2 + a_1 t + a_2 \leftrightarrow (a_0, a_1, a_2)$, the problem is translated into (2). The condition is $a \neq \frac{26}{3}$.

(5) The problem is the existence of x_1, x_2, x_3, such that

$$\begin{pmatrix} 10 & 11 \\ 11 & 12 \end{pmatrix} = x_1 \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 & a \\ a & 9 \end{pmatrix} = \begin{pmatrix} x_1 + 4x_2 + 7x_3 & 2x_1 + 5x_2 + ax_3 \\ 2x_1 + 5x_2 + ax_3 & 3x_1 + 6x_2 + 9x_3 \end{pmatrix}.$$

This is equivalent to the system

$$x_1 + 4x_2 + 7x_3 = 10, \quad 2x_1 + 5x_2 + ax_3 = 11, \quad 3x_1 + 6x_2 + 9x_3 = 12.$$

Then we are back to (1), and the condition is $a \neq 8$.

(6) Similar to (5), the problem is translated to (2), by $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \leftrightarrow (x, y, z)$. The condition is $a \neq \frac{26}{3}$.

Exercise 1.15

Applying $R_i \leftrightarrow R_j$ and $R_i \leftrightarrow R_j$ again, the i-th and j-th rows are changed as follows (the marks (i) and (j) indicate the i-th and j-th positions)

$$(\underset{(i)}{R_i}, \underset{(j)}{R_j}) \to (\underset{(i)}{R_j}, \underset{(j)}{R_i}) \to (\underset{(i)}{R_i}, \underset{(j)}{R_j}).$$

The other rows are not changed.

Applying cR_i and $c^{-1}R_i$, the i-th row is changed as follows

$$\underset{(i)}{R_i} \to \underset{(i)}{cR_i} \to \underset{(i)}{c^{-1}cR_i} = \underset{(i)}{R_i}.$$

The other rows are not changed.

Applying $R_i + cR_j$ and $R_i - cR_j$, the i-th and j-th rows are changed as follows

$$(\underset{(i)}{R_i}, \underset{(j)}{R_j}) \to (\underset{(i)}{R_i + cR_j}, \underset{(j)}{R_j}) \to (\underset{(i)}{(R_i + cR_j) - cR_j}, \underset{(j)}{R_j}) = (\underset{(i)}{R_i}, \underset{(j)}{R_j}).$$

The other rows are not changed.

Exercise 1.16

Suppose $x_1, \dots, x_n$ satisfy the i-th and j-th equations

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= p, \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= q. \end{aligned}$$

Then

1. They satisfy

$$\begin{aligned} b_1x_1 + b_2x_2 + \cdots + b_nx_n &= q, \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &= p. \end{aligned}$$

   This shows the solution is still solution after the first row operation.

2. They satisfy

$$\begin{aligned} ca_1x_1 + ca_2x_2 + \cdots + ca_nx_n &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = cp, \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= q. \end{aligned}$$

   This shows the solution is still solution after the second row operation.

3. They satisfy

$$\begin{aligned}(a_1+cb_1)x_1+(a_2+cb_2)x_2+\cdots+(a_n+cb_n)x_n &= (a_1x_1+a_2x_2+\cdots+a_nx_n)\\ &\quad + c(b_1x_1+b_2x_2+\cdots+b_nx_n) = p+cq,\\ b_1x_1+b_2x_2+\cdots+b_nx_n &= q.\end{aligned}$$

This shows the solution is still solution after the third row operation.

By Exercise 1.15, the row operations can be reversed, and the reverses are also row operations. Therefore row operations do not change the solutions.

Exercise 1.17 (1)

$$\begin{pmatrix}0 & \bullet & * & *\\ 0 & 0 & 0 & 0\\ \bullet & * & * & *\end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix}0 & \bullet & * & *\\ \bullet & * & * & *\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Exercise 1.17 (2)

For suitable c, we have

$$\begin{pmatrix}\bullet & * & * & *\\ \bullet & * & * & *\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{R_2+cR_1} \begin{pmatrix}\bullet & * & * & *\\ 0 & * & * & *\\ 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & 0 & 0\end{pmatrix} \text{ or } \begin{pmatrix}\bullet & * & * & *\\ 0 & 0 & \bullet & *\\ 0 & 0 & 0 & 0\end{pmatrix} \text{ or } \cdots.$$

Exercise 1.17 (3)

For suitable c, we have

$$\begin{aligned}\begin{pmatrix}0 & \bullet & * & *\\ 0 & \bullet & * & *\\ \bullet & * & * & *\end{pmatrix} &\xrightarrow[R_1\leftrightarrow R_2]{R_2\leftrightarrow R_3} \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & \bullet & * & *\end{pmatrix}\\ &\xrightarrow{R_3+cR_2} \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & \bullet & *\end{pmatrix} \text{ or } \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & 0 & \bullet\end{pmatrix} \text{ or } \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & 0 & 0\end{pmatrix}.\end{aligned}$$

Exercise 1.17 (4)

$$\begin{pmatrix}0 & \bullet & * & *\\ 0 & 0 & \bullet & *\\ \bullet & * & * & *\end{pmatrix} \xrightarrow[R_1\leftrightarrow R_2]{R_2\leftrightarrow R_3} \begin{pmatrix}\bullet & * & * & *\\ 0 & \bullet & * & *\\ 0 & 0 & \bullet & *\end{pmatrix}.$$

Exercise 1.18

2×2 row echelon forms

$$\begin{pmatrix}\bullet & *\\ 0 & \bullet\end{pmatrix} \begin{pmatrix}\bullet & *\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & \bullet\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

3 × 3 row echelon forms

$$\begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ 0 & 0 & \bullet \end{pmatrix} \begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bullet & * & * \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bullet & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bullet & * \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bullet & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 1.19

First we consider the $n \times n$ case. It can be easily found out that, after the pivot columns are chosen to be fixed, we are able to determine one unique row echelon form (the pivot of the first pivot column has to lie on the first row, and the pivot of the second pivot column has to lie on the second row, and so on). Therefore, the number of row echelon forms is equal to the number of ways to choose pivot columns, which is:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

When $m \geq n$, we can choose at most n pivot columns, so the case is the same as mentioned above, the number of row echelon forms is 2^n; when $m < n$, we should notice that we can only choose at most m pivot columns, thus the number of row echelon forms is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}.$$

Exercise 1.20

2 × 2 reduced row echelon forms

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3 × 3 reduced row echelon forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 1.21

(1) $x_1 = b_1 - a_1x_2$, $x_3 = b_2$, x_2 arbitrary.
(2) $x_1 = -a_1x_2 - b_1x_4$, $x_3 = -b_2x_4$, x_2, x_4 arbitrary.
(3) $x_1 = b_1$, $x_2 = b_2$, $x_3 = b_3$.
(4) $x_1 = b_1 - a_1x_2 - a_2x_3$, x_2, x_3 arbitrary.
(5) $x_1 = -a_1x_2 - a_2x_3 - b_1x_4$, x_2, x_3, x_4 arbitrary.
(6) $x_1 = b_1 - a_1x_2 - a_2x_4$, $x_3 = b_2 - a_3x_4$, x_2, x_4 arbitrary.
(7) $x_1 = b_1 - a_1x_3 - a_2x_4$, $x_2 = b_2 - a_3x_3 - a_4x_4$, x_3, x_4 arbitrary.
(8) $x_1 = b_1 - a_1x_3$, $x_2 = b_2 - a_2x_3$, x_3 arbitrary.
(9) $x_2 = b_1 - a_1x_4$, $x_3 = b_2 - a_2x_4$, x_1, x_3 arbitrary.

(10) $x_1 = b_1 - a_1x_3,\ x_2 = b_2 - a_2x_3,\ x_3$ arbitrary.

(11) $x_1 = b_1 - a_1x_3 - a_2x_5,\ x_2 = b_2 - a_3x_3 - a_4x_5,\ x_3 = b_3 - a_5x_5,\ x_3, x_5$ arbitrary.

(12) $x_1 = -a_1x_3 - a_2x_5 - b_1x_6,\ x_2 = -a_3x_3 - a_4x_5 - b_2x_6,\ x_3 = -a_5x_5 - b_3x_6,\ x_3, x_5, x_6$ arbitrary.

Exercise 1.22

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -5 & -6 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & -6 & 4 & 0 \end{pmatrix}$$

SECTION 1.3

EXERCISE 1.23

For a general matrix in $M_{3\times 2}$, we have

$$\begin{pmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows every vector in $M_{2\times 3}$ is a lienar combination of six matrices. Moreover, if the linear combination above is equals another linear combination

$$\begin{pmatrix} y_1 & y_4 \\ y_2 & y_5 \\ y_3 & y_6 \end{pmatrix} = y_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + y_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + y_5 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + y_6 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then we get the equality of matrices

$$\begin{pmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{pmatrix} = \begin{pmatrix} y_1 & y_4 \\ y_2 & y_5 \\ y_3 & y_6 \end{pmatrix}.$$

This means $x_1 = y_1, \dots, x_6 = y_6$. This shows the uniqueness of the coefficients in the linear combination of six matrices. Therefore the six matrices form a basis.

In general, a basis of $M_{m\times n}$ consists of $m \times n$ matrices, such that one entry is 1 and all the other entries are 0. There are mn such matrices.

EXERCISE 1.24

By $\vec{v}_i = 0\vec{v}_1 + \cdots + 0\vec{v}_{i-1} + 1\vec{v}_i + 0\vec{v}_{i+1} + \cdots + 0\vec{v}_n$, we have

$$[\vec{v}_i]_\alpha = (0, \dots, 0, \underset{(i)}{1}, 0, \dots, 0) = \vec{e}_i.$$

EXERCISE 1.25

We have

$$x_1\vec{v}_{\pi(1)} + x_2\vec{v}_{\pi(2)} + \cdots + x_n\vec{v}_{\pi(n)} = x_{\pi^{-1}(1)}\vec{v}_1 + x_{\pi^{-1}(2)}\vec{v}_2 + \cdots + x_{\pi^{-1}(n)}\vec{v}_n.$$

Therefore the coefficients are related by the inverse permutation. This does not change the existence and uniqueness of linear combination expression. Therefore $\pi(\alpha)$ is still a basis. Moreover, the equality shows

$$[\vec{x}]_{\pi(\alpha)} = (x_1, x_2, \dots, x_n) \iff [\vec{x}]_\alpha = (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \dots, x_{\pi^{-1}(n)}) = \pi^{-1}([\vec{x}]_{\pi(\alpha)}).$$

EXERCISE 1.26

We have $x(1,2) + y(2,4) = (x + 2y, 2x + 4y) = (a, 2a)$, where $a = x + 2y$. Therefore $(1,1)$ is not a linear combination of $(1,2)$ and $(2,4)$.

Exercise 1.27

If $(c,d) \neq (0,0)$, then $ad = bc$ implies $(a,b) = \lambda(c,d)$ for some λ. Then $x(a,b) + y(c,d) = (\lambda x + y)(c,d)$. We can easily find a vector that is not a scalar multiple of (c,d). Then the vector is not a linear combination of (a,b) and (c,d).

Exercise 1.28 (1)

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & 0 \\ 0 & -5 & -7 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -18 & 0 \end{pmatrix}$$

Since all rows are pivot, the four column vectors span $\mathbb{R}^3$.

Exercise 1.28 (2)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -18 \\ 0 & 0 & 0 \end{pmatrix}$$

Since not all rows are pivot, the three column vectors do not span $\mathbb{R}^4$.

Exercise 1.28 (3)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since not all rows are pivot, the three column vectors do not span $\mathbb{R}^3$.

Exercise 1.28 (4)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & a \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a-5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & a-6 \end{pmatrix}$$

All rows are pivot if and only if $a \neq 6$, which is the condition for the four column vectors to span $\mathbb{R}^3$.

Exercise 1.28 (5)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & a \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a-5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & a-6 \\ 0 & 0 & 0 \end{pmatrix}$$

Since not all rows are pivot, the three column vectors do not span $\mathbb{R}^4$.

Exercise 1.28 (6)

$$\begin{pmatrix} 0 & 2 & -1 & 4 \\ -1 & 3 & 0 & 1 \\ 2 & -4 & -1 & 2 \\ 1 & 1 & -2 & 7 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 0 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 4 \\ 0 & 4 & -2 & 8 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 0 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since not all rows are pivot, the four column vectors do not span $\mathbb{R}^4$.

EXERCISE 1.28 (7)

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since not all rows are pivot, the four column vectors do not span $\mathbb{R}^4$.

EXERCISE 1.28 (8)

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & a \\ 0 & 1 & a & b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & a-1 \\ 0 & 0 & a-1 & b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & b-(a-1)^2 \end{pmatrix}$$

All rows are pivot if and only if $b \neq (a-1)^2$, which is the condition for the four column vectors to span $\mathbb{R}^4$.

EXERCISE 1.29

If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V, then any vector $\vec{x} \in V$ is a linear combination of the vectors

$$\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + 0\vec{w}.$$

The right side is also a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$.

Note: Exercises 1.29, 1.30, 1.31 can also be p roved by using Proposition 1.2.1, without calculation.

EXERCISE 1.30

Suppose $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$.

If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$ span V, then any vector $\vec{x} \in V$ is a linear combination of the vectors

$$\begin{aligned} \vec{x} &= x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + y\vec{w} \\ &= x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + y(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) \\ &= (x_1 + ya_1)\vec{v}_1 + (x_2 + ya_2)\vec{v}_2 + \cdots + (x_n + ya_n)\vec{v}_n. \end{aligned}$$

The right side is also a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Therefore $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ also span V.

The converse follows from Exercise 1.29.

EXERCISE 1.31

By Exercise 1.29, we know $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ spanning V implies $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ spanning V. Then by $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ being linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, we may apply Exercise 1.30 one by one. We find $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ spanning V implies $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ spanning V.

Exercise 1.32

$1 \iff 2$ is trivial.

For $1 \implies 3$, if any vector is a linear combination of $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n$, then any vector $\vec{x} \in V$ is

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_i\vec{v}_i + \cdots + x_n\vec{v}_n = x_1\vec{v}_1 + \cdots + \frac{x_i}{c}c\vec{v}_i + \cdots + x_n\vec{v}_n.$$

Therefore any $\vec{x} \in V$ is a linear combination of $\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n$.

For $1 \implies 4$, if any vector is a linear combination of $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n$, , then any vector $\vec{x} \in V$ is

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_i\vec{v}_i + \cdots + x_j\vec{v}_j + \cdots + x_n\vec{v}_n = x_1\vec{v}_1 + \cdots + x_i(\vec{v}_i + c\vec{x}_j) + \cdots + (x_j - cx_i)\vec{v}_j + \cdots + x_n\vec{v}_n.$$

Therefore any $\vec{x} \in V$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_i + c\vec{x}_j, \dots, \vec{v}_j, \dots, \vec{v}_n$,

For the converse, we know that $3 \implies 1$ is $1 \implies 3$ with c^{-1} in place of c, and $4 \implies 1$ is $1 \implies 4$ with $-c$ in place of c.

Exercise 1.33

If $m > n$, then n vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$. Moreover, if A is an $m \times n$ matrix, then $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$.

Exercise 1.34

(1) 3 vectors cannot span $\mathbb{R}^5$. This means the following system of linear equations has no solution for some right side

$$\begin{aligned}
10x_1 \qquad\quad + 8x_3 &= b_1,\\
-2x_1 + 8x_2 - 9x_3 &= b_2,\\
3x_1 - 2x_2 + 3x_3 &= b_3,\\
7x_1 + 5x_2 + 6x_3 &= b_4,\\
2x_1 - 4x_2 + 5x_3 &= b_5.
\end{aligned}$$

(2) 4 vectors cannot span $\mathbb{R}^5$. This means the following system of linear equations has no solution for some right side

$$\begin{aligned}
10x_1 \qquad\quad + 8x_3 + 7x_4 &= b_1,\\
-2x_1 + 8x_2 - 9x_3 - 9x_4 &= b_2,\\
3x_1 - 2x_2 + 3x_3 + 3x_4 &= b_3,\\
7x_1 + 5x_2 + 6x_3 - 5x_4 &= b_4,\\
2x_1 - 4x_2 + 5x_3 + 6x_4 &= b_5.
\end{aligned}$$

(3) The linear combination of the five vectors is always $(0, *, *, *, *)$. Therefore $(1, 0, 0, 0, 0)$ is not a linear combination of the five vectors. This means the following system of linear

equations has no solution

$$\begin{aligned} 0 &= 1, \\ -2x_1 + 8x_2 - 9x_3 - 5x_4 + 4x_5 &= 0, \\ 3x_1 - 2x_2 + 3x_3 + 4x_4 - x_5 &= 0, \\ 7x_1 + 5x_2 + 6x_3 + 2x_4 + 3x_5 &= 0, \\ 2x_1 - 4x_2 + 5x_3 - 7x_4 - 6x_5 &= 0. \end{aligned}$$

(4) The linear combination of the five vectors is always $(2a, *, a, *, *)$. Therefore $(1, 0, 1, 0, 0)$ is not a linear combination of the five vectors. This means the following system of linear equations has no solution

$$\begin{aligned} 6x_1 - 4x_2 + 6x_3 + 8x_4 - 2x_5 &= 1, \\ -2x_1 + 8x_2 - 9x_3 - 5x_4 + 4x_5 &= 0, \\ 3x_1 - 2x_2 + 3x_3 + 4x_4 - x_5 &= 1, \\ 7x_1 + 5x_2 + 6x_3 + 2x_4 + 3x_5 &= 0, \\ 2x_1 - 4x_2 + 5x_3 - 7x_4 - 6x_5 &= 0. \end{aligned}$$

Exercise 1.35

(1) Since not all columns are pivot, the four vectors are linearly dependent.

(2) Since all columns are pivot, the three vectors are linearly independent.

(3) Since not all columns are pivot, the three vectors are linearly dependent.

(4) Since not all columns are pivot, the four vectors are linearly dependent.

(5) All columns are pivot if and only if $a \neq 6$, which is the condition for the three vectors to be linearly independent.

(6) Since not all columns are pivot, the four vectors are linearly dependent.

(7) Since not all columns are pivot, the four vectors are linearly dependent.

(5) All columns are pivot if and only if $b \neq (a-1)^2$, which is the condition for the four vectors to be linearly independent.

Exercise 1.36

$1 \iff 2$ is trivial.

For $1 \implies 3$, suppose $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n$ are linearly independent. Then

$$x_1\vec{v}_1 + \cdots + x_i(c\vec{v}_i) + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + \cdots + y_i(c\vec{v}_i) + \cdots + y_n\vec{v}_n$$

means

$$x_1\vec{v}_1 + \cdots + cx_i\vec{v}_i + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + \cdots + cy_i\vec{v}_i + \cdots + y_n\vec{v}_n.$$

By $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n$ linearly independent, this implies

$$x_1 = y_1, \dots, cx_i = cy_i, \dots, x_n = y_n.$$

By $c \neq 0$, this is the same as

$$x_1 = y_1, \dots, x_i = y_i, \dots, x_n = y_n.$$

This verifies $\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n$ are linearly independent.

For $1 \implies 4$, suppose $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n$ are linearly independent. Then

$$x_1\vec{v}_1 + \cdots + x_i(\vec{v}_i + c\vec{x}_j) + \cdots + x_j\vec{v}_j + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + \cdots + y_i(\vec{v}_i + c\vec{v}_i) + \cdots + y_j\vec{v}_j + \cdots + y_n\vec{v}_n$$

means

$$x_1\vec{v}_1 + \cdots + x_i\vec{v}_i + \cdots + (cx_i + x_j)\vec{v}_j + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + \cdots + y_i\vec{v}_i + \cdots + (cy_i + y_j)\vec{v}_j + \cdots + y_n\vec{v}_n$$

By $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n$ linearly independent, this implies

$$x_1 = y_1, \dots, x_i = y_i, \dots, cx_i + x_j = cy_i + y_j, \dots, x_n = y_n.$$

This is the same as

$$x_1 = y_1, \dots, x_i = y_i, \dots, x_j = y_j, \dots, x_n = y_n.$$

This verifies $\vec{v}_1, \dots, \vec{v}_i + c\vec{v}_j, \dots, \vec{v}_j, \dots, \vec{v}_n$ are linearly independent.

For $1 \implies 4$, if any vector is a linear combination of $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n$, , then any vector $\vec{x} \in V$ is

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_i\vec{v}_i + \cdots + x_j\vec{v}_j + \cdots + x_n\vec{v}_n = x_1\vec{v}_1 + \cdots + x_i(\vec{v}_i + c\vec{x}_j) + \cdots + (x_j - cx_i)\vec{v}_j + \cdots + x_n\vec{v}_n.$$

Therefore any $\vec{x} \in V$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_i + c\vec{x}_j, \dots, \vec{v}_j, \dots, \vec{v}_n$,

For the converse, we know that $3 \implies 1$ is $1 \implies 3$ with c^{-1} in place of c, and $4 \implies 1$ is $1 \implies 4$ with $-c$ in place of c.

Exercise 1.37

If $m < n$, then n vectors in $\mathbb{R}^m$ are linearly dependent. Moreover, if A is an $m \times n$ matrix, then the solution of $A\vec{x} = \vec{b}$ is not unique.

Exercise 1.38 (1)

The 6 vectors in $\mathbb{R}^3$ are always linearly dependent. This means the solution of the following system is not unique

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 + 3x_5 + 2x_6 &= b_1, \\ 2x_1 + 3x_2 + x_3 + 3x_4 + 2x_5 + x_6 &= b_2, \\ 3x_1 + 1x_2 + 2x_3 + 2x_4 + x_5 + 3x_6 &= b_3. \end{aligned}$$

Exercise 1.38 (2)

The 5 vectors in $\mathbb{R}^4$ are always linearly dependent. This means the solution of the following system is not unique

$$\begin{aligned} x_1 + 10x_2 \qquad\quad + 8x_4 + 7x_5 &= b_1, \\ 3x_1 - 2x_2 + 8x_3 - 9x_4 - 9x_5 &= b_2, \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 + 3x_5 &= b_3, \\ -4x_1 + 7x_2 + 5x_3 + 6x_4 - 5x_5 &= b_4. \end{aligned}$$

Exercise 1.38 (3)

The 4 vectors in $\mathbb{R}^4$ are linearly dependent, because the last vector is π multiple of the first. This means the solution of the following system is not unique

$$\begin{aligned} x_1 + 10x_2 \qquad\quad + \pi x_4 &= b_1, \\ 3x_1 - 2x_2 + 8x_3 + 3\pi x_4 &= b_2, \\ 2x_1 + 3x_2 - 2x_3 + 2\pi x_4 &= b_3, \\ -4x_1 + 7x_2 + 5x_3 - 4\pi x_4 &= b_4. \end{aligned}$$

Exercise 1.38 (4)

The 5 vectors in $\mathbb{R}^5$ are linearly dependent. The reason is that the last row of the associated 5×5 matrix is all 0. Therefore the matrix has at most 4 pivots. Therefore not all five columns are pivot. The linearly dependene means the solution of the following system is not unique

$$\begin{aligned} x_1 + 10x_2 \qquad\quad + 8x_4 + 7x_5 &= b_1, \\ 3x_1 - 2x_2 + 8x_3 - 9x_4 - 9x_5 &= b_2, \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 + 3x_5 &= b_3, \\ -4x_1 + 7x_2 + 5x_3 + 6x_4 - 5x_5 &= b_4 \\ 0 &= b_5. \end{aligned}$$

Exercise 1.39

Suppose $a > b$ and $xe^{at} + ye^{bt} = 0$. Then $x + ye^{(b-a)t} = 0$. By $b - a < 0$ and taking $\lim_{t\to+\infty}$m we get $x = 0$. Then we have $ye^{bt} = 0$. This implies $y = 0$. By Proposition 1.3.7, this proves e^{at} and e^{bt} are linearly independent.

Suppose a, b, c are distinct. Then we may assume $a > b > c$. If $xe^{at} + ye^{bt} + ze^{ct} = 0$. Then $x + ye^{(b-a)t} + ze^{(c-a)t} = 0$. By $b - a < 0$, $c - a < 0$, and taking $\lim_{t\to+\infty}$m we get $x = 0$. Then $xe^{at} + ye^{bt} + ze^{ct} = 0$ becomes $ye^{bt} + ze^{ct} = 0$. We are back to the linear independence of two functions e^{bt} and e^{ct}, which we already proved.

Exercise 1.40

Assume that $x_1 \cos t + x_2 \sin t + x_3 e^t = 1$. If $x_3 \neq 0$, then let t goes to ∞, then we have $\infty = 1$, which is a contradiction. Thus $x_3 = 0$. And then let $t = 0, \pi$, we have $x_1 = 1, x_1 = -1$, which is a contradiction.

Exercise 1.41 (1)

Suppose $x \cos^2 t + y \sin^2 t = 0$. Taking $t = 0$, we get $x = 0$. Taking $t = \frac{\pi}{2}$, we get $y = 0$. The two functions are linearly independent.

We have $1 = \cos^2 t + \sin^2 t$. Therefore 1 is a linear combination of $\cos^2 t, \sin^2 t$.

Suppose $x \cos^2 t + y \sin^2 t = t$. Taking $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$, we get $x = 1, \frac{1}{2}x + \frac{1}{2}y = \frac{\pi}{4}, y = \frac{\pi}{4}$. Then we get $\frac{1}{2} + \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{2}$, a contradiction. Therefore t is not a linear combination of $\cos^2 t, \sin^2 t$.

Exercise 1.41 (2)

By $\cos^2 t + \sin^2 t - 1 = 0$, the three functions are linearly dependent.

We have $\cos 2t = \cos^2 t - \sin^2 t$. Therefore $\cos 2t$ is a linear combination of $\cos^2 t, \sin^2 t$.

Since 1 is a linear combination of $\cos^2 t, \sin^2 t$, by Exercise 1.30, we know t is a linear combination of $\cos^2 t, \sin^2 t, 1$ if and only if it is a linear combination of $\cos^2 t, \sin^2 t$. By part (1), however, t t is not a linear combination of $\cos^2 t, \sin^2 t$. Therefore t is not a linear combination of $\cos^2 t, \sin^2 t, 1$.

Exercise 1.41 (3)

Suppose $x_1 + x_2 t + x_3 e^t + x_4 te^t = 0$. Taking $t = 0$, we get $x_1 = 0$. Therefore $x_2 t + x_3 e^t + x_4 te^t = 0$. This implies $x_2 + x_3 t^{-1} e^t + x_4 e^t = 0$. Taking $\lim_{t\to+\infty}$, we get $x_2 = 0$. Therefore $x_3 e^t + x_4 te^t = 0$. This implies $x_3 + x_4 t = 0$. Taking $t = 0$, we get $x_3 = 0$. There3fore $x_4 t = 0$. Taking $t = 1$, we get $x_4 = 0$. This shows $1, t, e^t, te^t$ are linearly independent.

By $(1+t)e^t = 01 + 0t + 1e^t + 1te^t$, we know $(1+t)e^t$ is a linear combination of $1, t, e^t, te^t$.

By $((1+t)e^t)' = 2e^t + te^t$, we know $((1+t)e^t)'$ is a linear combination of $1, t, e^t, te^t$.

Exercise 1.41 (4)

Suppose $x\cos^2 t + y\cos 2t = 0$. Taking $t = \frac{1}{2}\pi$, we get $-y = 0$, and $y = 0$. Therefore $x\cos^2 t = 0$. Taking $t = 0$, we get $x = 0$. The two functions are linearly independent.

By $\cos 2t = 2\cos^2 t - 1$, we have $1 = 2\cos^2 t - \cos^2 t$. Then $a = 2a\cos^2 t - a\cos^2 t$ is a linear combination of $\cos^2 t, \cos 2t$.

By $a + \sin^2 t = a + 1 - \cos^2 t = (a+1)(2\cos^2 t - \cos^2 t) - \cos^2 t = 2(a+1)\cos^2 t - (a+2)\cos^2 t$, we find $a + \sin^2 t$ is a linear combination of $\cos^2 t, \cos 2t$.

Exercise 1.42

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$ are linearly independent. By Proposition 1.3.8, we know $\vec{w}$ is not a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Moreover, to see the linear independence of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, we consider

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n.$$

This is the same as

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + 0\vec{w} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n + 0\vec{w}.$$

Since $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$ are linearly independent, this implies

$$x_1 = y_1,\ x_2 = y_2,\ \dots, x_n = y_n,\ 0 = 0.$$

This proves that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.

Conversely, suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent, and $\vec{w}$ is not a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. For the linear independence of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$, we use Proposition 1.3.7 and consider

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + x_{n+1}\vec{w} = \vec{0}.$$

If $x_{n+1} \neq 0$, then we have

$$\vec{w} = -\frac{x_1}{x_{n+1}}\vec{v}_1 - \frac{x_2}{x_{n+1}}\vec{v}_2 + \cdots - \frac{x_n}{x_{n+1}}\vec{v}_n.$$

This contradicts the assumption that $\vec{w}$ is not a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Therefore $x_{n+1} = 0$, and we get

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}.$$

Then by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ linearly independent, we get $x_1 = x_2 = \cdots = x_n = 0$. Together with $x_{n+1} = 0$, this proves $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{w}$ are linearly independent.

Exercise 1.43

Suppose $\vec{v}_i$ is a linear $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}$. Then $\vec{v}_i$ is also a linear $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$. By Proposition 1.3.8, this implies $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent.

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent. Then by Proposition 1.3.8, we have

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0},$$

in which some $x_i \neq 0$. Let i be the biggest index, such that $x_i \neq 0$. Then we have

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_i\vec{v}_i = \vec{0}.$$

This implies

$$\vec{v}_i = -\frac{x_1}{x_i}\vec{v}_1 - \frac{x_2}{x_i}\vec{v}_2 + \cdots - \frac{x_{i-1}}{x_i}\vec{v}_{i-1}.$$

Therefore $\vec{v}_i$ is a linear combination of the previous vectors.

Exercise 1.44

Since all four rows are pivot, by Proposition 1.3.4, the six columns span $\mathbb{R}^4$.

Applying the same row operations to the 1st, 3rd, 4th, 6th columns, we get

$$(\vec{v}_1, \vec{v}_3, \vec{v}_4, \vec{v}_6) \to \begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \end{pmatrix}.$$

We find all four row pivot. Therefore the four vectors still span $\mathbb{R}^4$.

Applying the same row operations to the 1st, 3rd, 4th columns, we get

$$(\vec{v}_1, \vec{v}_3, \vec{v}_4) \to \begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}.$$

We find the last row not pivot. In fact, if we apply the same row operations to any three columns, we get at most three pivots. Therefore the last row can never be pivot. Therefore the three vectors cannot span $\mathbb{R}^4$.

Exercise 1.45 (1)

By $x_0 + x_1t + x_2t^2 + x_3t^3 \leftrightarrow (x_0, x_1, x_2, x_3)$, the vectors become the column vectors of a matrix, and we carry out row operation

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{pmatrix}$$

Since all rows are pivot, the six polynomials span P_3. A minimal spanning set is given by the first four columns, corresponding to $1+t, 1+t^2, 1+t^3, t+t^2$.

Exercise 1.45 (2)

By $x_0 + x_1 t + x_2 t^2 + x_3 t^3 \leftrightarrow (x_0, x_1, x_2, x_3)$, the vectors become the column vectors of a matrix, and we carry out row operation

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 2 & -2 & 2 & 0 & 0 & 2 \\ 3 & -3 & 3 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 & -2 & 0 \\ 0 & 0 & 3 & 3 & 0 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & -3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Since all rows are pivot, the six polynomials span P_3. A minimal spanning set is given by the 1st, 3rd, 5th, 6th columns, corresponding to $t+2t^2+3t^3, 1+2t^2+3t^3, 1+t+3t^3, 1+t+2t^2$.

Exercise 1.46 (1)

Linearly independent subset is given by first three columns $(1,2,3), (2,3,1), (3,1,2)$. They already form a basis of $\mathbb{R}^3$.

Exercise 1.46 (2)

Linearly independent subset is given by the three columns $(1,2,3,1), (2,3,1,2), (3,1,2,3)$. The following shows adding $(0,0,0,1)$ gives a basis of $\mathbb{R}^4$.

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.46 (3)

Linearly independent subset is given by first two columns $(1,2,3), (2,3,4)$. They do not form a basis of $\mathbb{R}^3$. The following shows adding $(0,0,1)$ gives a basis of $\mathbb{R}^3$.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.46 (4)

Linearly independent subset is given by $(1,2,3), (2,3,4)$ in case $a = 6$, and given by $(1,2,3), (2,3,4), (4,5,a)$ in case $a \neq 6$. In case $a = 6$, by (3), adding $(0,0,1)$ gives a basis of $\mathbb{R}^3$. In case $a \neq 6$, $(1,2,3), (2,3,4), (4,5,a)$ is a basis of $\mathbb{R}^3$.

Exercise 1.46 (5)

Linearly independent subset is given by $(1,2,3,4),(2,3,4,5)$ in case $a=6$, and given by $(1,2,3,4),(2,3,4,5),(3,4,5,a)$ in case $a\neq 6$.

In case $a\neq 6$, the following shows adding $(0,0,1,0)$ to $(1,2,3,4),(2,3,4,5),(3,4,5,a)$ gives a basis of $\mathbb{R}^4$ (note that adding $(0,0,0,1)$ does not give a basis).

$$\begin{pmatrix}1&2&3&0\\2&3&4&0\\3&4&5&1\\4&5&a&0\end{pmatrix}\to\begin{pmatrix}1&2&3&0\\1&1&1&0\\1&1&1&1\\1&1&a-5&0\end{pmatrix}\to\begin{pmatrix}1&1&1&0\\0&1&2&0\\0&0&a-6&0\\0&0&0&1\end{pmatrix}$$

In case $a=6$, by taking $a=7\neq 6$, we know adding $(3,4,5,7),(0,0,1,0)$ to $(1,2,3,4),(2,3,4,5)$ gives a basis of $\mathbb{R}^4$.

Exercise 1.46 (6)

Linearly independent subset is given by $(0,-1,2,1),(2,3,-4,1)$. The following shows adding $(-1,0,1,0),(4,1,0,1)$ (modifications of original columns 3 and 4) to $(0,-1,2,1),(2,3,-4,1)$ gives a basis of $\mathbb{R}^4$.

$$\begin{pmatrix}0&2&-1&4\\-1&3&0&1\\2&-4&1&0\\1&1&0&1\end{pmatrix}\to\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&2&1&2\\0&4&0&2\end{pmatrix}\to\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&0&2&-2\\0&0&2&-6\end{pmatrix}\to\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&0&2&-2\\0&0&0&-4\end{pmatrix}$$

Exercise 1.46 (7)

Linearly independent subset is given by $(1,0,1,0),(0,1,0,1),(0,1,1,0)$. The following shows adding $(1,0,0,0)$ gives a basis of $\mathbb{R}^4$.

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&1&0\\0&1&0&0\end{pmatrix}\to\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&-1\\0&0&-1&0\end{pmatrix}\to\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

Exercise 1.46 (8)

Linearly independent subset is given by $(1,0,1,0),(0,1,0,1),(0,1,1,0)$ in case $b=(a-1)^2$, and given by $(1,0,1,0),(0,1,0,1),(1,0,a,b)$ in case $b\neq(a-1)^2$. In case $b\neq(a-1)^2$, the four vectors form a basis of $\mathbb{R}^4$. By taking $a=0$ and $b=1$, adding $(1,0,-1,1)$ to $(1,0,1,0),(0,1,0,1),(0,1,1,0)$ gives a basis of $\mathbb{R}^4$.

Exercise 1.47

Suppose $x_1t^2(t-1)+x_2t(t^2-1)+x_3(t^2-4)=0$. Taking $t=0$, we get $-4x_3=0$. Therefore $x_3=0$, and $x_1t^2(t-1)+x_2t(t^2-1)=0$. This implies $x_1t(t-1)+x_2(t^2-1)=0$ for $t\neq 0$. Taking $\lim_{t\to 0}$, we get $-x_2=0$. Therefore $x_2=0$, and $x_1t(t-1)=0$. Then we get $x_1=0$. By Proposition 1.3.7, this verifies $t^2(t-1),t(t^2-1),t^2-4$ are linearly independent.

To extend to a basis of P_3, we add a polynomial that is not a linear combination of $t^2(t-1),t(t^2-1),t^2-4$. Suppose $1=x_1t^2(t-1)+x_2t(t^2-1)+x_3(t^2-4)$. By taking $t=0,1$, we get $1 = -4x_3$ and $1 = -3x_3$. This is a contradiction. Therefore 1 is not a linear combination of the three polynomials. Then $t^2(t-1), t(t^2-1), t^2-4, 1$ are linearly independent and form a basis of P_3.

Exercise 1.48

(1) 3 polynomials cannot span the 4 dimensional P_3.

(2) 3 matrices cannot span the 4 dimensional $M_{2\times 2}$

(3) The four matrices are of the form $\begin{pmatrix} a & * \\ * & 2a \end{pmatrix}$. Their linear combination is also of the form $\begin{pmatrix} a & * \\ * & 2a \end{pmatrix}$. Therefore the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not their linear combinations. The four matrices do not span $M_{2\times 2}$.

Exercise 1.49 Explain that the vectors are linearly dependent.

1. $3 + \sqrt{2}t - \pi t^2$, $e + 100t + 2\sqrt{3}t^2$, $4\pi t - 15.2t^2$, $\sqrt{\pi} + e^2t^2$.
2. $\begin{pmatrix} 3 & 8 \\ 4 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 8 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$.
3. $\begin{pmatrix} \pi & \sqrt{3} \\ 1 & 2\pi \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \pi \\ -10 & 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 3 & 100 \\ -77 & 6 \end{pmatrix}, \begin{pmatrix} \sin 2 & \pi \\ \sqrt{2}\pi & 2\sin 2 \end{pmatrix}$.

Exercise 1.50

Suppose α spans V. By Theorem 1.3.10, α contains a basis α' of V. Then $\#\alpha \geq \#\alpha' = \dim V$.

Suppose α is a linearly independent set V. By Theorem 1.3.11, α can be enlarged to a basis α' of V. Then $\#\alpha \leq \#\alpha' = \dim V$.

Exercise 1.51

Under the asumption the number of vectors in α is $\dim V$, by Theorem 1.3.14, we know (1) α spans V, and (2) α is linearly independent are equivalent. Then it is a simple logic that, if (1) $\iff$ (2), then (1) $\iff$ (2) $\iff$ (1 + 2). We note that (3) is (1+2).

Exercise 1.52 (1)

We have row operation

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since all row and columns are pivot, the three vectors form a basis.

Exercise 1.52 (2)

We have row operation

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

Since all row and columns are pivot, the three vectors form a basis.

Exercise 1.52 (3)

We have row operation

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}\to\begin{pmatrix}3&3&3&3\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}\to\begin{pmatrix}1&1&1&1\\0&0&-1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\to\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

Since all row and columns are pivot, the three vectors form a basis.

Exercise 1.53 (1)

Suppose $x_1(1+t)+x_2(1+t^2)+x_3(t+t^2)=0$. Taking $t=-1$, we get $x_2=0$. Then $x_1(1+t)+x_3(t+t^2)=0$. Compare coefficient of t^2, we get $x_3=0$. Then $x_1(1+t)=0$. This implies $x_3=0$. By Proposition 1.3.7, the three polynomials are linearly independent. By $\dim P_2=3$ and Theorem 1.3.15 (also see Exercise 1.51), the three vectors form a basis of P_2.

Alternatively, the problem can be translated to Exercise 1.52 (1).

Exercise 1.53 (2)

By $\begin{pmatrix}a&b\\c&d\end{pmatrix}\leftrightarrow(a,b,c,d)$, the problem is translated to Exercise 1.52 (2). The four mattices form a basis.

Exercise 1.54 (1)

We have row operation

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&a\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&-1&1\\0&1&a\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&-1&1\\0&0&a+1\end{pmatrix}.$$

The three vectors form a basis if and only if $a\neq-1$.

Exercise 1.54 (2)

We have row operation

$$\begin{pmatrix}1&1&0\\-1&0&1\\0&-1&a\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&1&1\\0&-1&a\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&-1&1\\0&0&a+1\end{pmatrix}.$$

The three vectors form a basis if and only if $a\neq-1$.

Exercise 1.54 (3)

We have row operation

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&a\end{pmatrix}\to\begin{pmatrix}1&1&1&0\\0&0&-1&1\\0&-1&0&1\\0&1&1&a\end{pmatrix}\to\begin{pmatrix}1&1&1&0\\0&-1&0&1\\0&0&-1&1\\0&0&0&a+2\end{pmatrix}$$

The three vectors form a basis if and only if $a \neq -2$.

Exercise 1.55

(1) Translated into Exercise 1.54 (1), basis if and only if $a \neq -1$.
(2) Translated into Exercise 1.54 (3), basis if and only if $a \neq -2$.

Exercise 1.56

If $a \neq 0$, then we have row operation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \to \begin{pmatrix} a & c \\ 0 & d - \frac{b}{a}c \end{pmatrix}$$

The two vectors form a basis if and only if $d - \frac{b}{a}c \neq 0$, i.e., $ad \neq bc$.

If $b \neq 0$, then we have row operation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \to \begin{pmatrix} b & d \\ a & c \end{pmatrix} \to \begin{pmatrix} b & d \\ 0 & c - \frac{a}{b}d \end{pmatrix}$$

The two vectors form a basis if and only if $c - \frac{a}{b}d \neq 0$, i.e., $ad \neq bc$.

If $a = b = 0$, then the two vectors do not form a basis.

Alternatively, the two vectors do not form a basis if and only if the two are linearly dependent. By Example 1.3.13, this means they are parallel, and is equivalent to $ad = bc$. Therefore the two vectors form a basis, i.e.e, linearly independent, if and only if $ad \neq bc$.

Exercise 1.57

All columns and all rows are pivot. This means the reduced row echelon form is the identity matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Exercise 1.58

We use α-coordinate to identify the vector space with Euclidean space. Then β is a basis if and only if $[\beta]_\alpha$ is a basis.

We have row operations

$$[\beta]_\alpha = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix},$$

$$[\beta]_\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$[\beta]_\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$[\beta]_\alpha = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

All rows and all columns are pivot. Therefore $[\beta]_\alpha$ is a basis. This implies β is a basis.

Exercise 1.59

By Exercises 1.32, the span properties of the four vector sets are equivalent. By Exercises 1.36, the liner independence properties of the four vector sets are equivalent. Since basis means the span property and the liner independence property, the basis properties of the four vector sets are also equivalent.

Exercise 1.60 (1)

By $(x_1, x_2) = x_2(0,1) + x_1(1,0)$, we have $[(x_1, x_2)]_\alpha = (x_2, x_1)$.

Exercise 1.60 (2)

By

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{pmatrix}$$

We get

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]_\alpha = x_1[\vec{e}_1]_\alpha + x_2[\vec{e}_2]_\alpha = x_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Exercise 1.60 (3)

By $(x_1, x_2) = \frac{x_1}{a}(a, 0) + \frac{x_2}{b}(0, b)$, we have $[(x_1, x_2)]_\alpha = (\frac{x_1}{a}, \frac{x_2}{b})$.

Exercise 1.60 (4)

By $\cos^2\theta + \sin^\theta = 1$, we have

$$(1,0) = \cos\theta(\cos\theta, \sin\theta) - \sin\theta(-\sin\theta, \cos\theta),$$
$$(0,1) = \sin\theta(\cos\theta, \sin\theta) + \cos\theta(-\sin\theta, \cos\theta).$$

Then

$$\begin{aligned}(x_1,x_2) &= x_1(1,0)+x_2(0,1)\\ &= x_1(\cos\theta(\cos\theta,\sin\theta)-\sin\theta(-\sin\theta,\cos\theta))\\ &+ x_2(\sin\theta(\cos\theta,\sin\theta)+\cos\theta(-\sin\theta,\cos\theta))\\ &= (x_1\cos\theta+x_2\sin\theta)(\cos\theta,\sin\theta)+(-x_1\sin\theta+x_2\cos\theta)(-\sin\theta,\cos\theta)\end{aligned}$$

Therefore
$$\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]_\alpha = \begin{pmatrix}x_1\cos\theta+x_2\sin\theta\\ -x_1\sin\theta+x_2\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Exercise 1.60 (5)

We have

$$\begin{pmatrix}1&0&0&1&0&0\\2&1&0&0&1&0\\3&2&1&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&1&0&0\\0&1&0&-2&1&0\\0&2&1&-3&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&1&0&0\\0&1&0&-2&1&0\\0&0&1&1&-2&1\end{pmatrix}$$

Then
$$\left[\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right]_\alpha = \begin{pmatrix}1&0&0\\-2&1&0\\1&-2&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Exercise 1.60 (6)

This permutes the order of basis vectors in (6). Then we do the same permutation of the columns
$$\left[\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right]_\alpha = \begin{pmatrix}0&0&1\\1&0&-2\\-2&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Exercise 1.60 (7)

We have

$$\begin{pmatrix}-1&0&0&1&0&0\\1&1&1&0&1&0\\2&1&0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&-1&0&0\\0&1&1&1&1&0\\0&1&0&2&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&-1&0&0\\0&1&0&2&0&1\\0&0&1&-1&1&-1\end{pmatrix}$$

Then
$$\left[\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right]_\alpha = \begin{pmatrix}-1&0&0\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Exercise 1.60 (8)

We have

$$\begin{pmatrix} 0 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & -3 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 4 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -4 & -2 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 & -\frac{3}{2} & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 2 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 & \frac{5}{2} & -\frac{5}{4} & 2 & -\frac{23}{4} \\ 0 & 2 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & -\frac{5}{4} & \frac{3}{2} & -\frac{19}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

Then

$$\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right]_\alpha = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{3}{2} & -\frac{19}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 2 & -1 & 1 & -3 \\ \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Exercise 1.61 (1)

We have

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then

$$\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right]_\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Exercise 1.61 (2)

We have

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then

$$\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right]_\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Exercise 1.61 (3)

We have

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -1 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -1 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & -1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Then

$$\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right]_\alpha = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Exercise 1.62 (1)

We form the matrix with the given vectors as column vectors. Then we do $R_1 + R_2, R_2 + R_3, \ldots$ and get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Since the last row is not pivot, the vectors do not form a basis.

In fact, by $(\vec{e}_1 - \vec{e}_2) + (\vec{e}_2 - \vec{e}_3) + \cdots + (\vec{e}_{n-1} - \vec{e}_n) + (\vec{e}_n - \vec{e}_1) = \vec{0}$, the vectors are linearly dependent.

Exercise 1.62 (2)

We form the matrix with the givne vectors as column vectors. Then we do $R_2 + R_1, R_3 + R_2, \ldots$ and get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & 0 & 1 & \cdots & 0 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}.$$

The vectors form a basis. The row operation

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 2 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & n-1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & n & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ 0 & 1 & \cdots & 0 & 0 & \frac{n-2}{n} & \frac{n-2}{n} & \cdots & -\frac{2}{n} & -\frac{2}{n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & -\frac{n-1}{n} \\ 0 & 0 & \cdots & 0 & 1 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \end{pmatrix}$$

We get

$$[\vec{x}]_\alpha = \begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ \frac{n-2}{n} & \frac{n-2}{n} & \cdots & -\frac{2}{n} & -\frac{2}{n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & -\frac{n-1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \end{pmatrix} \vec{x} = \frac{1}{n} \begin{pmatrix} n-1 & -1 & \cdots & -1 & -1 \\ n-2 & n-2 & \cdots & -2 & -2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & -(n-1) \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \vec{x}.$$

Exercise 1.62 (3)

We form the matrix with the given vectors as column vectors. Then we do $R_2 - R_1, R_3 - R_2, \dots$ and get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & (-1)^n \\ 0 & 0 & 0 & \cdots & 0 & 1-(-1)^n \end{pmatrix}.$$

The vectors form a basis if and only if $1 - (-1)^n = 0$, or n is odd. In fact, if $n = 2k$ is even, then we have

$$\begin{aligned} &(\vec{e}_1 + \vec{e}_2) + (\vec{e}_3 + \vec{e}_4) + \cdots + (\vec{e}_{2k-3} + \vec{e}_{2k-2}) + (\vec{e}_{2k-1} + \vec{e}_{2k}) \\ =&(\vec{e}_2 + \vec{e}_3) + (\vec{e}_4 + \vec{e}_5) + \cdots + (\vec{e}_{2k-2} + \vec{e}_{2k-1}) + (\vec{e}_{2k} + \vec{e}_1). \end{aligned}$$

This shows the vectors are linearly dependent.

For odd n, the row operation

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & -1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 2 & 1 & -1 & \cdots & -1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \cdots & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \cdots & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & \frac{1}{2} & -\frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We get

$$[\vec{x}]_\alpha = \frac{1}{2} \begin{pmatrix} \color{red}{1} & \color{blue}{1} & \color{green}{-1} & \cdots & -1 & 1 & -1 \\ \color{green}{-1} & \color{red}{1} & \color{blue}{1} & \cdots & 1 & -1 & 1 \\ 1 & \color{green}{-1} & \color{red}{1} & \cdots & -1 & 1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & -1 & 1 & \cdots & \color{red}{1} & \color{blue}{1} & \color{green}{-1} \\ -1 & 1 & -1 & \cdots & \color{green}{-1} & \color{red}{1} & \color{blue}{1} \\ 1 & -1 & 1 & \cdots & 1 & \color{green}{-1} & \color{red}{1} \end{pmatrix} \vec{x}.$$

The matrix consists of 1-diagonals and (-1)-diagonals, and multiplied by $\frac{1}{2}$.

Exercise 1.62 (4)

The matrix formed by

$$\begin{aligned} \vec{v}_1 &= \vec{e}_1, \\ \vec{v}_2 &= \vec{e}_1 + 2\vec{e}_2, \\ \vec{v}_3 &= \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3, \\ &\vdots \\ \vec{v}_n &= \vec{e}_1 + 2\vec{e}_2 + \cdots + n\vec{e}_n, \end{aligned}$$

is already a row echelon form, with all rows and columns pivot. The vectors from a basis.

We have $\vec{e}_1 = \vec{v}_1$, $\vec{e}_2 = \frac{1}{2}(\vec{v}_2 - \vec{v}_1)$, $\vec{e}_3 = \frac{1}{3}(\vec{v}_3 - \vec{v}_3)$, $\ldots$, $\vec{e}_n = \frac{1}{n}(\vec{v}_n - \vec{v}_{n-1})$. Therefore

$$\begin{aligned} \vec{x} &= x_1\vec{v}_1 + x_2\tfrac{1}{2}(\vec{v}_2 - \vec{v}_1) + x_3\tfrac{1}{3}(\vec{v}_3 - \vec{v}_2) + \cdots + x_n\tfrac{1}{n}(\vec{v}_n - \vec{v}_{n-1}) \\ &= (x_1 - \tfrac{1}{2}x_2)\vec{v}_1 + (\tfrac{1}{2}x_2 - \tfrac{1}{3}x_3)\vec{v}_2 + (\tfrac{1}{3}x_3 - \tfrac{1}{4}x_4)\vec{v}_3 + \cdots + +(\tfrac{1}{n-1}x_{n-1} - \tfrac{1}{n}x_n)\vec{v}_{n-1} + \tfrac{1}{n}x_n\vec{v}_n. \end{aligned}$$

Therefore

$$[\vec{x}]_\alpha = \begin{pmatrix} x_1 - \frac{1}{2}x_2 \\ \frac{1}{2}x_2 - \frac{1}{3}x_3 \\ \frac{1}{3}x_3 - \frac{1}{4}x_4 \\ \vdots \\ \frac{1}{n-1}x_{n-1} - \frac{1}{n}x_n \\ \frac{1}{n}x_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n-1} & -\frac{1}{n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \vec{x}$$

Exercise 1.63 (1)

Same as Exercise 1.62 (1).

Exercise 1.63 (2)

Same as Exercise 1.62 (3).

Exercise 1.63 (3)

We have

$$\begin{aligned} & x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n \\ =& (x_0 - x_1 - x_2 - \cdots - x_n) + x_1(1+t) + x_2(1+t^2) + \cdots + x_n(1+t^n). \end{aligned}$$

Then

$$[x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n]_{1,1+t,1+t^2,\ldots,1+t^n} = (x_0 - x_1 - x_2 - \cdots - x_n, x_1, x_2, \ldots, x_n).$$

Exercise 1.63 (4)

We have

$$t^k = (1+(t-1))^k = \binom{k}{0} + \binom{k}{1}(t-1) + \binom{k}{2}(t-1)^2 + \cdots + \binom{k}{k}(t-1)^k.$$

This implies

$$[x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n]_{1,t-1,(t-1)^2,\ldots,(t-1)^n} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n-1}{0} & \binom{n}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n-1}{1} & \binom{n}{1} \\ 0 & 0 & \binom{2}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} & \binom{n}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{n}{n} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Exercise 1.64

For any (x_1, x_2), we ty to solve $y_1(a,b) + y_2(c,d) = (x_1, x_2)$. This is a system

$$ay_1 + cy_2 = x_1, \quad by_2 + dy_2 = x_2,$$

in variables y_1, y_2.

Multiply the first equation by b, multiply the second equation by a, and subtract, we get $(bc-ad)y_2 = bx_1 - ax_2$. By $ad \neq bc$, we get $y_2 = \frac{-bx_1+ax_2}{ad-bc}$. Similarly, we have $y_1 = \frac{dx_1-cx_2}{ad-bc}$. Therefore

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]_{(a,b),(c,d)} = \begin{pmatrix} \frac{dx_1-cx_2}{ad-bc} \\ \frac{-bx_1+ax_2}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$