SECTION 2.1

Exercise 2.1

(1) Linear transformation.

$$\begin{aligned} L(a(x_1,x_2,x_3)+b(y_1,y_2,y_3)) &= L(ax_1+by_1, ax_2+by_2, ax_3+by_3) \\ &= (ax_1+by_1, (ax_2+by_2)+(ax_3+by_3)) \\ &= a(x_1, x_2+x_3)+b(y_1, y_2+y_3) \\ &= aL(x_1,x_2,x_3)+bL(y_1,y_2,y_3). \end{aligned}$$

(2) Not linear transformation. The following two results are not equal.

$$\begin{aligned} L(0,1,1) &= (0, 1\cdot 1) = (0,1), \\ L(0,1,0)+L(0,0,1) &= (0,1\cdot 0)+(0,0\cdot 1) = (0,0). \end{aligned}$$

(3) Linear transformation.

$$\begin{aligned} L(a(x_1,x_2,x_3)+b(y_1,y_2,y_3)) &= L(ax_1+by_1, ax_2+by_2, ax_3+by_3) \\ &= (ax_3+by_3, ax_1+by_1, ax_2+by_2) \\ &= a(x_3,x_1,x_2)+b(y_3,y_1,y_2) \\ &= aL(x_1,x_2,x_3)+bL(y_1,y_2,y_3). \end{aligned}$$

(4) Linear transformation.

$$\begin{aligned} L(a(x_1,x_2,x_3)+b(y_1,y_2,y_3)) &= L(ax_1+by_1, ax_2+by_2, ax_3+by_3) \\ &= (ax_1+by_1)+2(ax_2+by_2)+3(ax_3+by_3) \\ &= a(x_1, 2x_2, 3x_3)+b(y_1, 2y_2, 3y_3) \\ &= aL(x_1,x_2,x_3)+bL(y_1,y_2,y_3). \end{aligned}$$

Exercise 2.2

(1) Not linear transformation. $L(2) = 2^2 = 4$ and $L(1)+L(1) = 1^2+1^2 = 2$ are not equal.

(2) Linear transformation. $L(af+bg) = (af+bg)(t^2) = af(t^2)+bg(t^2) = aL(f)+bL(g)$.

(3) Linear transformation. $L(af+bg) = (af+bg)''(t^2) = af''+bg'' = aL(f)+bL(g)$.

(4) Linear transformation. $L(af+bg) = (af+bg)(t-2) = af(t-2)+bg(t-2) = aL(f)+bL(g)$.

(5) Linear transformation. $L(af+bg) = (af+bg)(2t) = af(2t)+bg(2t) = aL(f)+bL(g)$.

(6) Linear transformation. $L(af+bg) = (af+bg)'+2(af+bg)(t^2) = af'+bg'+2af+2bg = a(f'+2f)+b(g'+2g) = aL(f)+bL(g)$.

(7) Linear transformation. $L(af+bg) = ((af+bg)(0)+(af+bg)(1), (af+bg)(2)) = (af(0)+bg(0)+af(1)+bg(1), af(2)+bg(2)) = a(f(0)+f(1), f(2))+b(g(0)+g(1), g(2)) = aL(f)+bL(g)$.

(8) Not linear transformation. $L(2) = 2\cdot 2 = 4$ and $L(1)+L(1) = 1\cdot 1+1\cdot 1 = 2$ are not equal.

(9) Linear transformation. $L(af+bg)=\int_0^1(af(t)+bg(t))dt = a\int_0^1 f(t)dt + b\int_0^1 g(t)dt = aL(f)+bL(g)$.

(10) Linear transformation. $L(af+bg)=\int_0^t \tau(af(\tau)+bg(\tau))d\tau = a\int_0^t \tau f(\tau)d\tau+b\int_0^t \tau g(\tau)d\tau = aL(f)+bL(g)$.

Exercise 2.3

The two statements are contrapositive of each other. We only need to prove the second statement. Suppose $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ are linearly independent. Then

$$\begin{aligned} & x_1\vec{v}_1+\cdots+x_n\vec{v}_n = y_1\vec{v}_1+\cdots+y_n\vec{v}_n \\ \implies & x_1L(\vec{v}_1)+\cdots+x_nL(\vec{v}_n) = y_1L(\vec{v}_1)+\cdots+y_nL(\vec{v}_n) \\ \implies & x_1=y_1,\ x_2=y_2,\ \dots,\ x_n=y_n. \end{aligned}$$

The first $\implies$ is by applying L and using the lear property of L. The second $\implies$ is by the linear independence of $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$.

Exercise 2.4

We carry out column operations

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -2 \\ -2 & -3 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

The matrix of the linear transformation is $\begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$.

Exercise 2.5

Using the idea of Example 2.1.13, for $\vec{v}_1=(1,-1,0)$, $\vec{v}_2=(1,0,-1)$, $\vec{v}_3=(1,1,1)$, the reflection F satisfies $F(\vec{v}_1)=\vec{v}_1$, $F(\vec{v}_2)=\vec{v}_2$, $F(\vec{v}_3)=-\vec{v}_3$. Then by the calculation in Example 2.1.13, we get

$$\begin{aligned} F(\vec{e}_1) &= \tfrac{1}{3}\vec{v}_1+\tfrac{1}{3}\vec{v}_2+\tfrac{1}{3}(-\vec{v}_3) = (\tfrac{1}{3},-\tfrac{2}{3},-\tfrac{2}{3}), \\ F(\vec{e}_2) &= -\tfrac{2}{3}\vec{v}_1+\tfrac{1}{3}\vec{v}_2+\tfrac{1}{3}(-\vec{v}_3) = (-\tfrac{2}{3},\tfrac{1}{3},-\tfrac{2}{3}), \\ F(\vec{e}_3) &= \tfrac{1}{3}\vec{v}_1-\tfrac{2}{3}\vec{v}_2+\tfrac{1}{3}(-\vec{v}_3) = (-\tfrac{2}{3},-\tfrac{2}{3},\tfrac{1}{3}). \end{aligned}$$

We conclude that the matrix of F is

$$(F(\vec{e}_1)\ F(\vec{e}_2)\ F(\vec{e}_3)) = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

The matrix can also be obtained from $P=\frac{1}{2}(F+I)$.

Exercise 2.6

We carry out the column operations

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{v}_1 & \vec{v}_2 & \vec{0} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \frac{2}{3} & 1 & 1 \\ -\frac{1}{3} & -1 & 0 \\ -\frac{1}{3} & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Then matrix of the projection is the lower half $\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

Exercise 2.7

We carry out the same column operations as in Exercise 2.6

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{v}_1 & \vec{v}_2 & \vec{0} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \\ 4 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ \frac{8}{3} & 3 & 4 \\ \frac{7}{3} & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & -1 & 0 \\ 3 & -1 & 0 \\ \frac{8}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{7}{3} & \frac{5}{3} & -\frac{4}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \\ \frac{8}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{7}{3} & -\frac{5}{3} & \frac{4}{3} \end{pmatrix}.$$

Then matrix of L is the lower half $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ \frac{8}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{7}{3} & -\frac{5}{3} & \frac{4}{3} \end{pmatrix}$.

Exercise 2.8

By $a_0 + a_1 t + a_2 t^2 + a_3 t^3 \leftrightarrow (a_0, a_1, a_2, a_3)$, we identify P_3 with $\mathbb{R}^4$. Then the problem becomes Exercise 2.7. Then

$$\begin{aligned} L(x,y,z) &= x(2 + 3t + \tfrac{8}{3}t^2 + \tfrac{7}{3}t^3) + y(1 + 1t - \tfrac{1}{3}t^2 - \tfrac{5}{3}t^3) + z(-\tfrac{4}{3}t^2 + \tfrac{4}{3}t^3) \\ & (2x+y) + (3x+y)t + \tfrac{1}{3}(8x - y - 4)t^2 + \tfrac{1}{3}(7x - 5y + 4)t^3. \end{aligned}$$

Exercise 2.9

We have linear transformations

$$L_1(f) = f(0),\ L_2(f) = f',\ L_3(f) = \int_0^t f(\tau)d\tau \colon C^\infty \to C^\infty.$$

The Newton-Leibniz formula means $I = L_1 + L_3 \circ L_2$.

EXERCISE 2.10

For $A = (a_{ij})$ and $B = (b_{ij})$, we have $cA + dB = (ca_{ij} + db_{ij})$. Then the following verifies the trace is a liner function (transformation)

$$\begin{aligned}\text{tr}(cA + dB) &= (ca_{11} + db_{11}) + (ca_{22} + db_{22}) + \cdots + (ca_{nn} + db_{nn}) \\ &= c(a_{11} + a_{22} + \cdots + a_{nn}) + d(b_{11} + b_{22} + \cdots + b_{nn}) = c\text{tr}A + d\text{tr}B.\end{aligned}$$

Alternatively, for each i, j, we know the map $A \to a_{ij}$ of picking the (i, j)-entry is linear. Then the sum of the linear functions of picking (i, i)-entries is still linear.

Since A and A^T have the same diagonal entries a_{ii}, we have $\text{tr}A^T = \text{tr}A$.

EXERCISE 2.11

Denote the evaluation map by $E(L) = L(\vec{v})$. Then the following verifies E is linear

$$E(aL + bK) = (aL + bK)(\vec{v})) = aL(\vec{v}) + bK(\vec{v}) = aE(L) + bE(K).$$

Exercise 2.13 becomes the following: Let $L\colon V \to W$ be a linear transformation.

1. Prove that $(K_1 + K_2) \circ L = K_1 \circ L + K_2 \circ L$ and $(aL) \circ K = a(L \circ K)$.
2. Prove that $L^* = \cdot \circ L\colon \text{Hom}(W, U) \to \text{Hom}(V, U)$ is a linear transformation.
3. Prove that $I^* = I$, $(L + K)^* = L^* + K^*$, $(aL)^* = aL^*$ and $(L \circ K)^* = K^* \circ L^*$.
4. Prove that $L \mapsto L^*\colon \text{Hom}(V, W) \to \text{Hom}(\text{Hom}(W, U),\ \text{Hom}(V, U))$ is a linear transformation.

We do Exercises 2.12 and 2.13 together.

EXERCISE 2.12 (1)

Let $K_1, K_2\colon U \to V$ be linear transformations. Then for any $\vec{u} \in U$, we have

$$\begin{aligned}(L \circ (aK_1 + bK_2))(\vec{u}) &= L((aK_1 + bK_2)(\vec{u})) = L(aK_1(\vec{u}) + bK_2(\vec{u})) \\ &= aL(K_1(\vec{u})) + bL(K_2(\vec{u})) = a(L \circ K_1)(\vec{u}) + b(L \circ K_2)(\vec{u}) \\ &= (aL \circ K_1 + bL \circ K_2)(\vec{u}).\end{aligned}$$

EXERCISE 2.13 (1)

For any $\vec{v} \in V$, we have

$$\begin{aligned}((aK_1 + bK_2) \circ L)(\vec{v}) &= (aK_1 + bK_2)(L(\vec{v})) = aK_1(L(\vec{v})) + bK_2(L(\vec{v})) \\ &= a(K_1 \circ L)(\vec{v}) + b(K_2 \circ L)(\vec{v}) = (aK_1 \circ L + bK_2 \circ L)(\vec{v}).\end{aligned}$$

EXERCISE 2.12 (2)

By $L_* = L\circ$, the equality $L \circ (aK_1 + bK_2) = aL \circ K_1 + bL \circ K_2$ in Exercise 2.12(1) becomes $L_*(aK_1 + bK_2) = aL_*(K_1) + bL_*(K_2)$. This means L_* is a linear transformation.

Exercise 2.13 (2)

By $L^* = \circ L$, the equality $(aK_1 + bK_2) \circ L = aK_1 \circ L + bK_2 \circ L$ in Exercise 2.13(1) becomes $L^*(aK_1 + bK_2) = aL^*(K_1) + bL^*(K_2)$. This means L^* is a linear transformation.

Exercises 2.12 and 2.13 (3)

We have $I_*(K) = I \circ K = K$. This means $I_* = I$.

We also have $I^*(K) = K \circ I = K$. This means $I^* = I$.

The equality $L \circ (aK_1 + bK_2)) = aL \circ K_1 + bL \circ K_2$ in Exercise 2.12(1) is the same as $(aK_1+bK_2)^*(L) = aK_1^*(L)+bK_2^*(L) = (aK_1^*+bK_2^*)(L)$. This means $(aK_1+bK_2)^* = aK_1^*+bK_2^*$ in Exercise 2.13(3).

The equality $(aK_1 + bK_2) \circ L = aK_1 \circ L + bK_2 \circ L$ in Exercise 2.13(1) is the same as $(aK_1 + bK_2)_*(L) = aK_{1*}(L) + bK_{2*}(L) = (aK_{1*} + bK_{2*})(L)$. This means $(aK_1 + bK_2)_* = aK_{1*} + bK_{2*}$ in Exercise 2.12(3).

Let $L\colon V \to W$ and $K\colon U \to V$ be linear transformations. For any linear transformation $M\colon X \to U$, we have

$$(L \circ K)_*(M) = (L \circ K) \circ M = L \circ (K \circ M) = L_*(K_*(M)) = (L_* \circ K_*)(M).$$

Therefore $(L \circ K)_* = L_* \circ K_*$.

For any linear transformation $M\colon W \to X$, we have

$$(L \circ K)^*(M) = M \circ (L \circ K) = (M \circ L) \circ K = K^*(L^*(M)) = (K^* \circ L^*)(M).$$

Therefore $(L \circ K)^* = K^* \circ L^*$.

Exercise 2.12 (4)

The equality $(aK_1 + bK_2)_* = aK_{1*} + bK_{2*}$ in Exercise 2.12(3) means $K \mapsto K_*$ is a linear transformation.

Exercise 2.13 (4)

The equality $(aK_1 + bK_2)^* = aK^{1*} + bK^{2*}$ in Exercise 2.13(3) means $K \mapsto K^*$ is a linear transformation.

Exercise 2.14

Let $f\colon Y \to Z$ and $g\colon X \to Y$. Then for any $\phi\colon W \to X$, we have

$$(f \circ g)_*(\phi) = (f \circ g) \circ \phi = f \circ (g \circ \phi) = f_*(g_*(\phi)) = (f_* \circ g_*)(\phi).$$

This verifies $(f \circ g)_* = f_* \circ g_*$. For any $\phi\colon Z \to W$, we also have

$$(f \circ g)^*(\phi) = \phi \circ (f \circ g) = (\phi \circ f) \circ g = g^*(f^*(\phi)) = (g^* \circ f^*)(\phi).$$

This verifies $(f \circ g)^* = g^* \circ f^*$.

Exercise 2.15

In Examples 2.1.10, we get the matrix of reflection

$$F_\rho = \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix}.$$

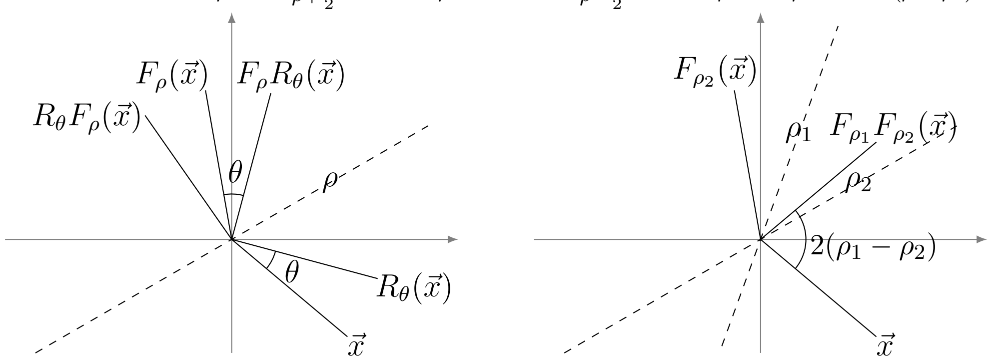
The equality

$$F_\rho^2 = \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix} \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2 2\rho + \sin^2 \sin 2\rho & 0 \\ 0 & \cos^2 2\rho + \sin^2 \sin 2\rho \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

means $\cos^2\theta + \sin^2\theta = 1$ for $\theta = 2\rho$.

Exercise 2.16

We have $R_\theta \circ F_\rho = F_{\rho+\frac{\theta}{2}}$, and $F_\rho \circ R_\theta = F_{\rho-\frac{\theta}{2}}$ and $F_{\rho_1} \circ F_{\rho_2} = R_{2(\rho_1-\rho_2)}$.



Then we get

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix} = \begin{pmatrix} \cos(2\rho+\theta) & \sin(2\rho+\theta) \\ \sin(2\rho+\theta) & -\cos(2\rho+\theta) \end{pmatrix},$$
$$\begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(2\rho-\theta) & \sin(2\rho-\theta) \\ \sin(2\rho-\theta) & -\cos(2\rho-\theta) \end{pmatrix},$$
$$\begin{pmatrix} \cos 2\rho_1 & \sin 2\rho_1 \\ \sin 2\rho_1 & -\cos 2\rho_1 \end{pmatrix} \begin{pmatrix} \cos 2\rho_2 & \sin 2\rho_2 \\ \sin 2\rho_2 & -\cos 2\rho_2 \end{pmatrix} = \begin{pmatrix} \cos 2(\rho_1-\rho_2) & \sin 2(\rho_1-\rho_2) \\ \sin 2(\rho_1-\rho_2) & -\cos 2(\rho_1-\rho_2) \end{pmatrix}.$$

The first equality means

$$\cos\theta\cos 2\rho - \sin\theta\sin 2\rho = \cos(2\rho+\theta),$$
$$\cos\theta\sin 2\rho + \sin\theta\cos 2\rho = \sin(2\rho+\theta).$$

The second equality means

$$\cos 2\rho\cos\theta + \sin 2\rho\sin\theta = \cos(2\rho+\theta),$$
$$\sin 2\rho\cos\theta - \cos 2\rho\sin\theta = \sin(2\rho-\theta).$$

The third equality means

$$\cos 2\rho_1\cos 2\rho_1 + \sin 2\rho_1\sin 2\rho_2 = \cos 2(\rho_1-\rho_2),$$
$$\sin 2\rho_1\cos 2\rho_1 - \cos 2\rho_1\sin 2\rho_2 = \sin 2(\rho_1-\rho_2).$$

Exercise 2.17

In Exercise 2.16, we see $R_\theta \circ F_\rho = F_{\rho+\frac{\theta}{2}}$, and $F_\rho \circ R_\theta = F_{\rho-\frac{\theta}{2}}$ are different. For example, if we take $\theta = \frac{1}{2}\pi$ and $\rho = \frac{1}{2}\pi$, then we get $R_{\frac{1}{2}\pi} \circ F_{\frac{1}{2}\pi} \neq F_{\frac{1}{2}\pi} \circ R_{\frac{1}{2}\pi}$, which means

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \neq \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 2.18

Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (b_{ij})$. Then

$$A + B = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = B + A,$$

and

$$(A + B) + C = ((a_{ij} + b_{ij}) + c_{ij}) = (a_{ij} + (b_{ij} + c_{ij})) = A + (B + C).$$

Exercise 2.20

We have $L \circ (K_1 + K_2) = L \circ K_1 + L \circ K_2$ and $L \circ (aK) = a(L \circ K)$ in Exercise 2.12(1). Let the matrices of L, K_1, K_2 be A, B, C, then $L \circ (K_1 + K_2) = L \circ K_1 + L \circ K_2$ becomes $A(B + C) = AB + AC$. Let the matrices of L, K be A, B, then $L \circ (aK) = a(L \circ K)$ becomes $A(aB) = a(AB)$.

Consider the transformation $L_A(X) = AX$ of matrices by multiplying A to the left. We have

$$L_A(aX + bY) = A(aX + bY) = A(aX) + A(bY) = a(AX) + b(AY) = aL_A(X) + bL_A(Y).$$

This shows L_A is a linear transformation.

Exercise 2.21

We have $(K_1 + K_2) \circ L = K_1 \circ L + K_2 \circ L$ and $(aK) \circ L = a(K \circ L)$ in Exercise 2.13(1). et the matrices of K_1, K_2, L be A, B, C, then $(K_1 + K_2) \circ L = K_1 \circ L + K_2 \circ L$ becomes $(A + B)C = AC + BC$. Let the matrices of K, L be A, B, then $(aK) \circ L = a(K \circ L)$ becomes $(aA)B = a(AB)$.

Explain that Exercise 2.13 means that the matrix multiplication satisfies $(A + B)C = AC + BC$, $(cA)B = c(AB)$, and the right multiplication $X \mapsto XA$ is a linear transformation.

Consider the transformation $R_A(X) = XA$ of matrices by multiplying A to the right. We have

$$R_A(aX + bY) = (aX + bY)A = (aX)A + (bY)A = a(XA) + b(YA) = aR_A(X) + bR_A(Y).$$

This shows R_A is a linear transformation.

Exercise 2.22

Using the notation in Exercises 2.20 and 2.21, we have $\mathrm{tr}AXB = (\mathrm{tr} \circ L_A \circ R_B)(X)$. By Exercises 2.10, 2.20, 2.21, we know tr, L_A, R_B are lienar transformations. Their composition is still a linear transformation.

Exercise 2.23

Let $A = (a_{ij})$ and $B = (b_{ji})$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then the (i, j)-entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Therefore

$$\operatorname{tr}(AB) = \sum_{i=1}^{m}(a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni}) = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}b_{ki}.$$

The (j, i)-entry of BA is $b_{j1}a_{1i} + b_{j2}a_{2i} + \cdots + b_{jm}a_{mi} = \sum_{k=1}^{m} b_{jk}a_{ki}$. Therefore

$$\operatorname{tr}(AB) = \sum_{j=1}^{n}(b_{j1}a_{1j} + b_{j2}a_{2j} + \cdots + b_{jm}a_{mj}) = \sum_{j=1}^{n}\sum_{k=1}^{m} b_{jk}a_{kj}.$$

By changing notations of the indices (i, k changed to k, j), we have

$$\sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}a_{ik} = \sum_{j=1}^{n}\sum_{k=1}^{m} b_{jk}a_{kj}.$$

Exercise 2.24

$$\begin{pmatrix}1&2\\3&4\end{pmatrix} + \begin{pmatrix}4&-2\\-3&1\end{pmatrix} = \begin{pmatrix}5&0\\0&5\end{pmatrix}, \qquad \begin{pmatrix}0&1&0\\1&0&1\end{pmatrix} + \begin{pmatrix}1&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&1&1\\1&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}4&-2\\-3&1\end{pmatrix} = \begin{pmatrix}-2&0\\0&-2\end{pmatrix}, \qquad \begin{pmatrix}4&-2\\-3&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix} = \begin{pmatrix}-2&0\\0&-2\end{pmatrix},$$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}2&1&2\\4&3&4\end{pmatrix}, \qquad \begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&2&1\\3&4&3\end{pmatrix},$$

$$\begin{pmatrix}4&-2\\-3&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}-2&4&-2\\1&-3&1\end{pmatrix}, \qquad \begin{pmatrix}4&-2\\-3&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}4&-2&4\\-3&1&-3\end{pmatrix},$$

$$\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix} = \begin{pmatrix}3&4\\1&2\\3&4\end{pmatrix}, \qquad \begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}4&-2\\-3&1\end{pmatrix} = \begin{pmatrix}-3&1\\4&-2\\-3&1\end{pmatrix},$$

$$\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}, \qquad \begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \begin{pmatrix}1&1&1\\2&2&2\end{pmatrix}, \qquad \begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \begin{pmatrix}2&2&2\\1&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&2\\1&2\\1&2\end{pmatrix}.$$

Exercise 2.25

(1) The matrix is rotation by θ. Applying the rotation n times is the rotation by $n\theta$. Therefore

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

(2) The matrix is reflection with respect to the line of angle θ. Applying the reflection even times is the identity

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Applying the reflection odd times is the reflection itself

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{2k+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(3) Direct calculation

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}^n = \begin{pmatrix} a_1^n & 0 & 0 & 0 \\ 0 & a_2^n & 0 & 0 \\ 0 & 0 & a_3^n & 0 \\ 0 & 0 & 0 & a_4^n \end{pmatrix}.$$

(4) Direct calculation

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^3 = \begin{pmatrix} 0 & 0 & 0 & a^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{>3} = O.$$

(5) Let the matrix in (4) be B, with a replaced by b. Then $A = aI + B$, and

$$\begin{aligned} A^2 &= a^2I + 2aB + B^2 \\ &= a^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2a\begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & b^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 & 0 \\ 0 & a^2 & 2ab & b^2 \\ 0 & 0 & a^2 & 2ab \\ 0 & 0 & 0 & a^2 \end{pmatrix}, \\ A^3 &= a^3I + 3a^2B + 3aB^2 + B^3 = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ 0 & a^3 & 3a^2b & 3ab^2 \\ 0 & 0 & a^3 & 3a^2b \\ 0 & 0 & 0 & a^3 \end{pmatrix}, \\ A^n &= a^nI + na^{n-1}B + \binom{n}{2}a^{n-2}B^2 + \binom{n}{3}a^{n-3}B^3 \\ &= \begin{pmatrix} a^n & na^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \binom{n}{3}a^{n-3}b^3 \\ 0 & a^n & na^{n-1}b & \binom{n}{2}a^{n-2}b^2 \\ 0 & 0 & a^n & na^{n-1}b \\ 0 & 0 & 0 & a^n \end{pmatrix}. \quad (B^n = O \text{ for } n > 3) \end{aligned}$$

Exercise 2.26

(1) We have row operations

$$\begin{pmatrix}1&2&3&7&8\\4&5&6&9&10\end{pmatrix}\to\begin{pmatrix}1&2&3&7&8\\0&-3&-6&-19&-22\end{pmatrix}\to\begin{pmatrix}1&0&-1&-\frac{17}{3}&-\frac{23}{3}\\0&1&2&\frac{19}{3}&\frac{22}{3}\end{pmatrix}.$$

We get

$$X=\begin{pmatrix}-\frac{17}{3}+x&-\frac{23}{3}+y\\\frac{19}{3}-2x&\frac{22}{3}-2y\\x&y\end{pmatrix}=\begin{pmatrix}-\frac{17}{3}&-\frac{23}{3}\\\frac{19}{3}&\frac{22}{3}\\0&0\end{pmatrix}+\begin{pmatrix}x&y\\-2x&-2y\\x&y\end{pmatrix}.$$

(2) We have row operations

$$\begin{pmatrix}1&4&7&10\\2&5&8&11\\3&6&a&b\end{pmatrix}\to\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&a-9&b-12\end{pmatrix}.$$

The solution exists if and only if $a=9$ and $b=12$ (both augmented matrices have solutions). The solution is

$$X=\begin{pmatrix}-1&-2\\2&3\end{pmatrix}.$$

(3) We have row operations

$$\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&b\end{pmatrix}\to\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&b-12\end{pmatrix}.$$

The solution exists if and only if $b=12$. The solution is

$$X=\begin{pmatrix}-1&-2\\2&3\end{pmatrix}.$$

Exercise 2.27

We have

$$AB=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}&a_{11}b_{12}+a_{12}b_{22}\\a_{21}b_{11}+a_{22}b_{21}&a_{21}b_{12}+a_{22}b_{22}\end{pmatrix},$$

$$B^TA^T=\begin{pmatrix}b_{11}&b_{21}\\b_{12}&b_{22}\end{pmatrix}\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}a_{11}+b_{21}a_{12}&b_{11}a_{21}+b_{21}a_{22}\\b_{12}a_{11}+b_{22}a_{12}&b_{12}a_{21}+b_{22}a_{22}\end{pmatrix}$$

Comparing the two, we get $(AB)^T=B^TA^T$.

(1) The equality $X\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is the same as $\begin{pmatrix}1&3\\2&4\end{pmatrix}X^T=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. We carry out row operation

$$\begin{pmatrix}1&3&1&0\\2&4&0&1\end{pmatrix}\to\begin{pmatrix}1&0&-2&\frac{3}{2}\\0&1&1&-\frac{1}{2}\end{pmatrix}.$$

Then

$$X^T = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}, \quad X = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

(2) First, we solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by row operation

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}, \quad Y = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then we solve $X \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = Y$ by taking the transpose $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} X^T = Y^T = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$. So we carry out row operation

$$\begin{pmatrix} 4 & -2 & -2 & \frac{3}{2} \\ -3 & 1 & 1 & -\frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 & -\frac{5}{4} \end{pmatrix}, \quad X^T = \begin{pmatrix} 0 & -\frac{1}{4} \\ 1 & -\frac{5}{4} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{5}{4} \end{pmatrix}.$$

Exercise 2.28

First, we need X to be 2×2 for AX and XA to have the same size. Let $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then

$$AX = \begin{pmatrix} -x_{11} & -x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad XA = \begin{pmatrix} -x_{11} & x_{12} \\ -x_{21} & x_{22} \end{pmatrix}.$$

Therefore $AX = XA$ if and only if $x_{12} = x_{21} = 0$. In other words, X is a *diagonal matrix*.

In general, if $a_1, a_2, \ldots, a_n$ are distinct in

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

then $AX = XA$ if and only if X is a diagonal matrix.

Exercise 2.29

$$T_{24} = T_{42} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_4(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$E_{35}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_{53}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 1 \end{pmatrix}.$$

Exercise 2.30

$T_{13}E_{13}(-2)D_2(3)A$ is a sequence of row operations

$$\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}\xrightarrow{3R_2}\begin{pmatrix}3&6&9\\6&15&24\\1&4&7\end{pmatrix}\xrightarrow{R_1-2R_3}\begin{pmatrix}1&-2&-5\\6&15&24\\1&4&7\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_3}\begin{pmatrix}1&4&7\\6&15&24\\1&-2&-5\end{pmatrix}.$$

$AT_{13}E_{13}(-2)D_2(3)$ is a sequence of column operations

$$\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}\xrightarrow{C_1\leftrightarrow C_3}\begin{pmatrix}7&4&1\\8&5&2\\9&6&3\end{pmatrix}\xrightarrow{C_1-2C_3}\begin{pmatrix}5&4&1\\4&5&2\\3&6&3\end{pmatrix}\xrightarrow{3C_2}\begin{pmatrix}5&12&1\\4&15&2\\3&18&3\end{pmatrix}.$$

Exercise 2.31

The equality $T_{ij}^2=I$ means the following (the rows should actually be vertical because each R_i is a row)

$$(\cdots R_i\cdots R_j\cdots)\xrightarrow{R_i\leftrightarrow R_j}(\cdots R_j\cdots R_i\cdots)\xrightarrow{R_i\leftrightarrow R_j}(\cdots R_i\cdots R_j\cdots).$$

The equality $D_i(a)D_i(b)=D_i(ab)$ means the following

$$(\cdots R_i\cdots)\xrightarrow{bR_i}(\cdots bR_i\cdots)\xrightarrow{aR_i}(\cdots abR_i\cdots).$$

The equality $E_{ij}(a)E_{ij}(b)=E_{ij}(a+b)$ means (we omit other rows $\cdots$)

$$(R_i,R_j)\xrightarrow{R_i+bR_j}(R_i+bR_j,R_j)\xrightarrow{R_i+aR_j}((R_i+bR_j)+aR_j,R_j)=(R_i+(a+b)R_j,R_j).$$

The equality $E_{ij}(a)=E_{ik}(a)E_{kj}(1)E_{ik}(a)^{-1}E_{kj}(1)^{-1}$ means the following

$$(R_i,R_j,R_k)\xleftarrow{R_k+1R_j}(R_i,R_j,R_k-R_j)\xleftarrow{R_i+aR_k}(R_i-a(R_k-R_j),R_j,R_k-R_j)$$
$$\xrightarrow{R_k+1R_j}(R_i-aR_k+aR_j,R_j,R_k)\xrightarrow{R_i+aR_k}(R_i+aR_j,R_j,R_k).$$

Exercise 2.32

(1) By

$$\begin{pmatrix}1&2&3&1\\2&3&1&2\\3&1&2&3\end{pmatrix}\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{pmatrix}1&2&3&1\\0&-1&-5&0\\0&-5&-7&0\end{pmatrix}\xrightarrow{R_3-5R_2}\begin{pmatrix}1&2&3&1\\0&-1&-5&0\\0&0&18&0\end{pmatrix},$$

we have

$$\begin{pmatrix}1&2&3&1\\2&3&1&2\\3&1&2&3\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\3&5&1\end{pmatrix}\begin{pmatrix}1&2&3&1\\0&-1&-5&0\\0&0&18&0\end{pmatrix}.$$

(2) By

$$\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\\1&2&3\end{pmatrix}\xrightarrow[R_4-R_1]{\substack{R_2-2R_1\\R_3-3R_1}}\begin{pmatrix}1&2&3\\0&-1&-5\\0&-5&-7\\0&0&0\end{pmatrix}\xrightarrow{R_3-5R_2}\begin{pmatrix}1&2&3\\0&-1&-5\\0&0&18\\0&0&0\end{pmatrix},$$

We have

$$\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\\1&2&3\end{pmatrix}=\begin{pmatrix}1&0&0&0\\2&1&0&0\\3&5&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1&2&3\\0&-1&-5\\0&0&18\\0&0&0\end{pmatrix}.$$

(3) By (different choice from the earlier exercise)

$$\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}\xrightarrow{\substack{R_2-2R_1\\R_3-3R_1}}\begin{pmatrix}1&2&3\\0&-1&-2\\0&-2&-4\end{pmatrix}\xrightarrow{R_3-2R_2}\begin{pmatrix}1&2&3\\0&-1&-2\\0&0&0\end{pmatrix},$$

we have

$$\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&-1&-2\\0&0&0\end{pmatrix}.$$

(6) By

$$\begin{pmatrix}0&2&-1&4\\-1&3&0&1\\2&-4&-1&2\\1&1&-2&7\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\2&-4&-1&2\\1&1&-2&7\end{pmatrix}\xrightarrow{\substack{R_3+2R_1\\R_4+R_1}}\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&2&-1&4\\0&4&-2&8\end{pmatrix}\xrightarrow{\substack{R_3-R_2\\R_4-2R_2}}\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

we have

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&2&-1&4\\-1&3&0&1\\2&-4&-1&2\\1&1&-2&7\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-2&1&1&0\\-1&-2&0&0\end{pmatrix}\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

or

$$\begin{pmatrix}0&2&-1&4\\-1&3&0&1\\2&-4&-1&2\\1&1&-2&7\end{pmatrix}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\-2&1&1&0\\-1&-2&0&0\end{pmatrix}\begin{pmatrix}-1&3&0&1\\0&2&-1&4\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

(7) By

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}\xrightarrow{R_4+R_3}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&-1\\0&0&0&0\end{pmatrix},$$

we have

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&-1\\0&0&0&0\end{pmatrix}.$$

Exercise 2.33

Column operation is row operation on A^T. The row operation on A^T gives $A^T = LU$. Then $A = U^T L^T$. Note that U^T is *lower triangular* (actually the column echelon form of A), and L^T is upper triangular. Therefore the column echelon form again gives LU-decomposition of A. The difference here is that L is the column echelon form, and have the same size as A, while U is square.