Eulerian 2-Strata Spaces

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Abstract

In [13], we introduced the notion of Eulerian stratified spaces. In this paper, we study in detail the topological and combinatorial properties of Eulerian 2-strata spaces, as the first step toward a deeper understanding of more strata cases. We show that Eulerian 2-strata spaces have similar topological structure as topological 2-strata spaces. Moreover, we find all the linear conditions on f-vectors over arbitrary abelian group.

1 Introduction

Stratification appears frequently and naturally in the study of spaces. The idea was already implicit in the earlier attempts at describing spaces in terms of simplicial complexes or regular cell complexes. Different versions of stratifications have been used to study the singularities of smooth maps, topologies of algebraic varieties, geometry of classes of maps, controlled topology, and group actions on manifolds. For example, Goresky and MacPherson invented intersection homology for pseudomanifolds [14] and generalized many classical theories on manifolds to more complicated spaces. They also developed stratified Morse theory [15] and found many applications to topologies of complex algebraic varieties and arrangement of linear subspaces. Cappell and Shaneson studied the characteristic classes in the intersection homologies of stratified spaces [9] [10] which, when applied to toric varieties, give the formula for counting lattice points in polytope. Quinn introduced the notion of homotopically stratified spaces [22], which play a central role in the study of group actions on topological manifolds.

The stratifications mentioned above are constructed mostly from purely topological consideration. In this paper and [13], we intend to study a new kind of stratification that has most relevance to the combinatorics of polyhedra. We hope that the similar philosophy may be applied (perhaps with different notion of stratification) to other combinatorial problems.

Topological stratification is to divide spaces into manifold pieces. The pieces should be glued together in a particularly nice way, so that the topological information about the manifold pieces can be patched together to give information on the whole space. The gluing may be understood in two steps: the relation between adjacent strata, and the compatibility of such relations at places where three or more strata meet.

The relation between strata is best understood through 2-strata spaces. This usually means a pair (X, Y), such that $X - Y$ and Y are manifolds, and a neighborhood of Y in X is homeomorphic to the mapping cylinder of a bundle $p_Y^X : L_Y^X = L \to Y$

with manifold fibre. Depending on the specific topological considerations, we have 2-strata spaces in various categories. The category that has the most relevance to us is the PL-category. This means that $X - Y$ and Y are PL-manifolds, and $L \to Y$ is a block bundle. This is our prototype of Eulerian 2-strata spaces.

The compatibility between relations is best understood through 3-strata spaces of the form $X \supset Y \supset Z$. The compatibility between three bundles $p_Y^X : L_Y^X \to Y$, $p_Z^X : L_Z^X \to Z$, $p_Z^Y : L_Z^Y \to Z$ roughly means that the "composition bundle" $p_Z^X(p_Y^X|_{L_Z^Y})$ is isomorphic to the "boundary bundle" ∂p_Z^X.

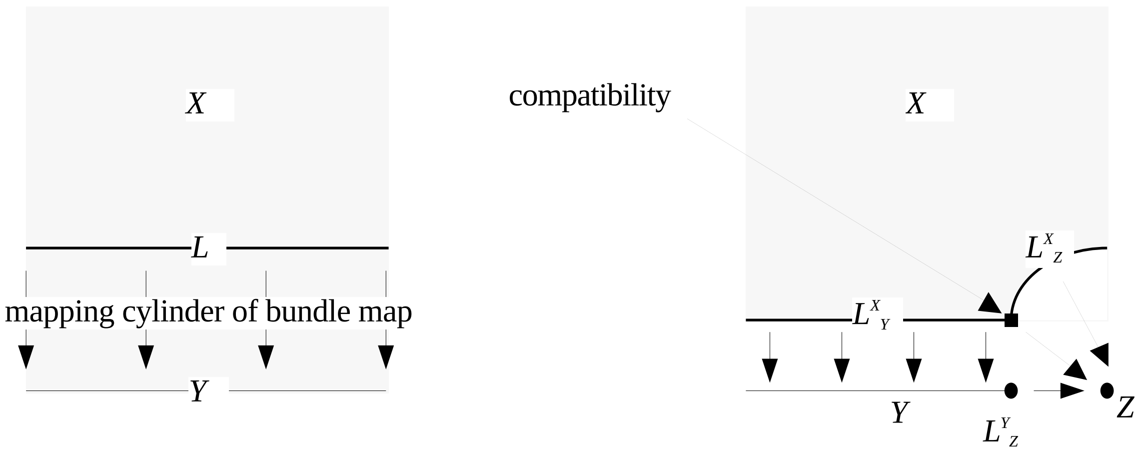

Figure 1: structure of 2-strata space and compatibility in 3-strata space

Combinatorially, V. Klee showed in [18] that the linear conditions on the f-vectors of simplicial complexes is closely related to the local Euler characteristic property of polyhedra. From this viewpoint, the classical linear combinatorics, especially the Dehn-Sommerville equations, is based on the assumption that local Euler characteristic data are constant throughout the polyhedra. This is comparable to topological manifolds, which may be considered as spaces with constant local topological data. Based on this analogy, we defined in [13] an Eulerian stratification to be a stratification in which local Euler characteristic data are constant over each stratum. In [13], we also generalized the classical result on *rational* linear relations on the f-vectors to Eulerian stratified spaces and studied the numerical relations between local Euler characteristic data in an Eulerian stratified space.

Although the notion of Eulerian stratification is motivated by linear conditions on f-vectors, it appears to us that Eulerian stratification has rich internal structure. This is already hinted by our numerical results in [13]. In fact, we hope the study of such internal structures may have implications on other combinatorial theories. Thus the first goal of this paper is to study the topological structure of Eulerian 2-strata spaces. We prove in Theorem 2.12 that Eulerian 2-strata spaces may be described in a way similar to topological 2-strata spaces. However, in place of bundle we should have a map $p : L \to Y$ between polyhedra such that the Euler characteristic $\chi(p^{-1}(y))$ is independent of the choice of $y \in Y$. In other words, p is an *Eulerian bundle* (bundle from Eulerian viewpoint). Furthermore, we give in Theorem 2.14 an explicit description of 2-dimensional Eulerian 2-strata spaces.

We do not know yet how the Eulerian bundles may fit together to describe the topological structure of general Eulerian stratified spaces. The situation is similar to the study of homotopically stratified spaces, introduced by Quinn in [22]. Homotopically stratified spaces are characterized by the the property that the local topological data is constant up to homotopy. In such spaces, one has to use certain homotopy version of bundle in place of bundle in the description of relations between adjacent strata. As a matter of fact, it is proved in [17] that homotopically stratified 2-strata spaces have *teardrop structures*. The compatibility between these teardrops in more strata case is yet to be worked out. Therefore if [13] is the Eulerian analogue of [22], then this paper is the Eulerian analogue of [17].

The second goal of this paper is to find linear conditions on f-vectors of Eulerian 2-strata spaces over any abelian group. The complete answer is explicitly given in Theorems 3.2, 3.3, and 3.4. We would like to emphasize that certain torsion linear conditions already appear for 2-strata case, while in the classical nonstratified case, all linear conditions are induced from integral ones (see [12], for example). We do not know yet all the torsion linear conditions for general Eulerian stratified spaces. We suspect they are related to the homology given by the boundary operation on the weight functions introduced in [13].

Some torsion linear conditions have appeared among the linear conditions for f-vectors of certain cubical complexes [1] [2] [7]. However, the complete answer is not yet known for these complexes, whereas in our 2-strata case we have a complete answer.

The classical theory on f-vectors of simplicial convex polytopes [4] [5] [16] [19] [20] [25] has been generalized in some other directions, notably on partially ordered sets [3] [6] [26] [27]. We expect that the idea of [12] [13] and this paper can be pursued in these directions. Moreover, we believe that the geometric Dehn-Sommerville relations [11] for simplicial polytopes (which implies the angle-sum relations of [21] [24] in particular) can also be generalized to more complicated geometric objects.

Throughout the paper all vectors are understood by columns, starting with the 0-th coordinate. For convenience, we often think of vectors in $\mathbf{Z}^n$ as vectors in $\mathbf{Z}^{n+l}$ by adding l zeros to the end.

We will use $\dot{A}$ to denote the interior of A. Thus we have the interior $\dot{\sigma}$ of a simplex σ, and the interior $\dot{M} = M - \partial M$ of a manifold M with boundary ∂M, etc.

χ denotes the Euler characteristic. Sometimes (especially in Section 2.4) we need to apply χ to spaces that are finite disjoint unions of interiors of simplices. For such spaces, χ is still the alternating sum of these simplices and has the usual properties one expects from Euler characteristic. In particular, $\chi(X_1 \sqcup X_2) = \chi(X_1) + \chi(X_2)$, $\chi(X_1 \times X_2) = \chi(X_1)\chi(X_2)$, and $\chi(\dot{D}^n) = (-1)^n$. For a vector $v = (v_0, v_1, v_2, \cdots)$, $\chi(v) = v_0 - v_1 + v_2 - \cdots$.

We will use many standard notations and results from piecewise linear topology. Our basic reference is [23].

2 Eulerian 2-Strata Spaces

2.1 Definitions

Let x be a point in a polyhedron X. Then x has a cone neighborhood xL with L compact and with x as the cone point (or apex). L is unique up to PL-homeomorphism, called the *link* of x in X, and denoted by $\mathrm{lk}(x, X)$. The closed cone neighborhood xL of x is called a *star* at x and denoted by $\mathrm{st}(x, X)$. The *open star* is the open cone neighborhood $\dot{\mathrm{st}}(x, X) = \mathrm{st}(x, X) - \mathrm{lk}(x, X)$.

A *polyhedron pair* (X, Y) is a polyhedron X with a subspace Y so that any point $x \in X$ has a cone neighborhood xL in X with $xL \cap Y = xK$ for some compact K. This implies in particular that Y is a polyhedron and is closed in X. We note that $K = \emptyset$ for $x \in X - Y$ and $K = \mathrm{lk}(x, Y)$ is the link in the subpolyhedron Y for $x \in Y$.

Definition 2.1 *A polyhedron pair* (X^n, Y) *is called an* Eulerian 2-strata space *if there is* $\beta \neq (-1)^{n-1}2$*, such that*

$$\chi(\mathrm{lk}(x, X)) = \begin{cases} 1 - (-1)^n & \text{for } x \notin Y \\ 1 + (-1)^n + \beta & \text{for } x \in Y. \end{cases}$$

We call β *the* local Euler characteristic. *We call* Y *the* lower stratum *and* $X - Y$ *the* upper stratum. *If* $\beta = (-1)^{n-1}$ *and* $\dim Y = n - 1$*, then we say* X *is an* Eulerian manifold with boundary Y *and write* $Y = \partial X$. *If* $Y = \emptyset$*, then we say* X *is a* boundaryless *Eulerian manifold.*

Remark 1 Later on we will show that Y is a boundaryless Eulerian manifold with dimension of different parity from n. Thus for $x \in Y$ we have $\chi(\mathrm{lk}(x, Y)) = 1 - (-1)^{n-1}$ and $\beta = \chi(\mathrm{lk}(x, X)) - \chi(\mathrm{lk}(x, Y))$, the Euler characteristic of the link of x in $X - Y$. This makes our terminology consistent with that of [13].

Remark 2 If $\beta = (-1)^{n-1}2$, then the whole space X is a boundaryless Eulerian manifold. Many results of this paper (especially the ones associated with Dehn-Sommerville equations) are still true in this case. However, some properties (especially the ones dealing with topological structures, such as Proposition 2.10 and Theorems 2.12 and 2.14) are not true in this case.

PL-manifolds with boundary are certainly Eulerian manifolds with boundary. If M^n is a boundaryless PL-manifold with $\chi(M) = 1 + (-1)^n$ (this is always the case if n is odd), then the cone cM is an $(n + 1)$-dimensional Eulerian manifold with boundary M. Moreover, wedges of even dimensional boundaryless PL-manifolds such as $X = S^2 \vee S^4$ are boundaryless Eulerian manifolds.

Wedges of odd dimensional boundaryless PL-manifolds such as $X = S^1 \vee S^3$ are Eulerian 2-strata spaces with the wedge point as the lower stratum. The disjoint unions of boundaryless PL-manifolds of various dimensions are also Eulerian 2-strata spaces. For example, $X = S^7 \sqcup S^2$ is an Eulerian 2-strata space with lower stratum $Y = S^2$ and relative Euler characteristic 0. In Sections 2.4 and 2.5, we give explicit descriptions of Eulerian 2-strata spaces.

Suppose Δ is a triangulation of a polyhedron pair (X, Y), i.e., Δ is a triangulation of X such that Y is a subcomplex with respect to the triangulation. Then for any simplex $\sigma \in \Delta$ we have the *simplicial link*

$$\mathrm{lk}(\sigma, \Delta) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \text{ and } \sigma \text{ span a simplex } \tau * \sigma \text{ of } \Delta\}.$$

The polyhedron

$$\mathrm{lk}(\sigma, X) = |\mathrm{lk}(\sigma, \Delta)|$$

is unique up to PL-homeomorphism, and is called the *link* of σ in X.

Let x be an interior point of σ. Then $\mathrm{lk}(x, X)$ is homeomorphic to the join $\partial\sigma * \mathrm{lk}(\sigma, X)$. Therefore

$$\begin{aligned} \chi(\mathrm{lk}(x, X)) &= \chi(\partial\sigma) + \chi(\mathrm{lk}(\sigma, X)) - \chi(\partial\sigma)\chi(\mathrm{lk}(\sigma, X)) \\ &= (1 - (-1)^{\dim\sigma}) + \chi(\mathrm{lk}(\sigma, X)) - (1 - (-1)^{\dim\sigma})\chi(\mathrm{lk}(\sigma, X)) \\ &= 1 - (-1)^{\dim\sigma} + (-1)^{\dim\sigma}\chi(\mathrm{lk}(\sigma, X)) \end{aligned}$$

and we conclude the following.

Proposition 2.2 *Let Δ be a triangulation of a polyhedron pair (X, Y). Then (X^n, Y) is an Eulerian 2-strata space with relative Euler characteristic β if and only if*

$$\chi(\mathrm{lk}(\sigma, X)) = \begin{cases} 1 - (-1)^{n-\dim\sigma} & \text{for } \sigma \not\subset Y \\ 1 + (-1)^{n-\dim\sigma} + (-1)^{\dim\sigma}\beta & \text{for } \sigma \subset Y. \end{cases}$$

In [18], Klee defined Eulerian manifolds as simplicial complexes such that the Euler characteristic of the link of each simplex is the same as the sphere in appropriate dimension. Proposition 2.2 implies that Klee's notion is the same as our notion of boundaryless Eulerian manifolds. However, our definition is more intrinsic because it does not depend on the choice of triangulations.

2.2 Dehn-Sommerville Equations

Let Δ be a triangulation of a polyhedron X. Its f-vector is

$$f(X;\Delta) = (f_0, f_1, f_2, \cdots), \quad f_i\text{=number of } i\text{-simplices in } \Delta.$$

In [18], Klee showed that the classical Dehn-Sommerville equations hold for his Eulerian manifolds. The proof started with the equality

$$\sum_{\dim \sigma = i} \chi(\mathrm{lk}(\sigma, X)) = \sum_{j>i} (-1)^{j-i-1} \binom{j+1}{i+1} f_j(X;\Delta), \tag{1}$$

which holds for all finite simplicial complexes (for a direct proof see [12]). If (X, Y) is a compact Eulerian 2-strata space with relative Euler characteristic β, then by Proposition 2.2,

$$\begin{aligned}\sum_{\dim \sigma = i} \chi(\mathrm{lk}(\sigma, X)) &= (1-(-1)^{n-i})(f_i(X;\Delta) - f_i(Y;\Delta)) + (1+(-1)^{n-i} + (-1)^i\beta) f_i(Y;\Delta) \\ &= (1-(-1)^{n-i}) f_i(X;\Delta) + (-1)^{n-i}(2+(-1)^n\beta) f_i(Y;\Delta). \end{aligned} \tag{2}$$

Combining (1) and (2) together, we obtain the following generalized Dehn-Sommerville equations.

Theorem 2.3 *Suppose Δ is a triangulation of a compact Eulerian 2-strata space (X^n, Y) with relative Euler characteristic β. Then for any $0 \le i \le n-1$,*

$$(1-(-1)^{n-i}) f_i(X;\Delta) + \sum_{j=i+1}^{n} (-1)^{n-j-1} \binom{j+1}{i+1} f_j(X;\Delta) = (2+(-1)^n\beta) f_i(Y;\Delta). \tag{3}$$

Let $D(n)$ denote the matrix of the coefficients on the left side of (3) (see [12] for explicit expression). Then (3) may be rewritten as

$$D(n) f(X;\Delta) = (2+(-1)^n\beta) f(Y;\Delta).$$

Since $\beta \neq (-1)^{n-1}2$ by assumption, the f-vector of the lower stratum is determined by the f-vector of the whole polyhedron. In [12], we showed that $D(n)$ has the following property

$$\chi(D(n)v) = (1-(-1)^n)\chi(v). \tag{4}$$

Applying this to (3), we obtain the following consequence of Theorem 2.3.

Corollary 2.4 *Suppose Δ is a triangulation of a compact Eulerian 2-strata space (X^n, Y) with relative Euler characteristic β. Then*

$$(2 + (-1)^n\beta)\chi(Y) = (1 - (-1)^n)\chi(X). \qquad (5)$$

Applying (5) to boundaryless Eulerian manifolds, we obtain the following result.

Corollary 2.5 *Suppose X is a compact odd dimensional Eulerian manifold without boundary. Then $\chi(X) = 0$.*

2.3 Properties of Eulerian 2-Strata Spaces

In this section we put together some properties of Eulerian manifolds and Eulerian 2-strata spaces.

Proposition 2.6 *$(X, Y) \times \mathbf{R}$ is an Eulerian 2-strata space with relative Euler characteristic β if and only if (X, Y) is an Eulerian 2-strata space with relative Euler characteristic $-\beta$.*

Proof: By $\mathrm{lk}((x, 0), X \times \mathbf{R}) = \mathrm{lk}(x, X) * \mathrm{lk}(0, \mathbf{R})$ and $\mathrm{lk}(0, \mathbf{R}) = \{2 \text{ points}\}$, we have

$$\chi(\mathrm{lk}((x, 0), X \times \mathbf{R})) = 2 - \chi(\mathrm{lk}(x, X)).$$

Then it is easy to see that the condition for $(X, Y) \times \mathbf{R}$ to be an Eulerian 2-strata space with relative Euler characteristic $-\beta$ is equivalent to the condition for (X, Y) to be an Eulerian 2-strata space with relative Euler characteristic β.

□

Proposition 2.7 *The product $(X_1, Y_1) \times (X_2, Y_2) = (X_1 \times X_2, X_1 \times Y_2 \cup Y_1 \times X_2)$ is an Eulerian manifold with boundary if and only if each pair is an Eulerian manifold with boundary.*

Proof: Suppose each pair is an Eulerian manifold. Then it follows from

$$\chi(\mathrm{lk}((x_1, x_2), X_1 \times X_2)) = \chi(\mathrm{lk}(x_1, X_1)) + \chi(\mathrm{lk}(x_2, X_2)) - \chi(\mathrm{lk}(x_1, X_1))\chi(\mathrm{lk}(x_2, X_2))$$

that the product is an Eulerian manifold with boundary. Conversely, suppose $(X_1, Y_1) \times (X_2, Y_2)$ is an Eulerian manifold. Then we take the interior U of a top dimensional simplex in some triangulation of X_1. As an open subset of $X_1 \times X_2$, $U \times (X_2, Y_2)$ is an Eulerian manifold. Since U is PL-homeomorphic to $\mathbf{R}^k$, we see that $\mathbf{R}^k \times (X_2, Y_2)$ is an Eulerian manifold. Then Proposition 2.6 inductively implies that (X_2, Y_2) is an Eulerian manifold.

$\square$

Lemma 2.8 *Suppose (X, Y) is an Eulerian 2-strata space with relative Euler characteristic β. Then*

1. *for $x \notin Y$, $\mathrm{lk}(x, X)$ is an Eulerian manifold without boundary;*
2. *for $x \in Y$, $(\mathrm{lk}(x, X), \mathrm{lk}(x, Y))$ is an Eulerian 2-strata space with relative Euler characteristic $-\beta$.*

Moreover, $\dim \mathrm{lk}(x, X)$ and $\dim X$ have different parity.

Lemma 2.9 *Suppose (X, Y) is an Eulerian 2-strata space with relative Euler characteristic β. Suppose σ is a simplex in a triangulation of (X, Y). Then*

1. *for $\sigma \not\subset Y$, $\mathrm{lk}(\sigma, X)$ is an Eulerian manifold without boundary;*
2. *for $\sigma \subset Y$, $(\mathrm{lk}(\sigma, X), \mathrm{lk}(\sigma, Y))$ is an Eulerian 2-strata space with relative Euler characteristic $(-1)^{\dim \sigma + 1}\beta$.*

Moreover, $\dim \mathrm{lk}(\sigma, X)$ and $\dim X - \dim \sigma$ have different parity.

Proof: We prove Lemma 2.9 first and then explain the modification needed for Lemma 2.8.

Suppose $\sigma \not\subset Y$ and $x \in \dot{\sigma}$. Then from PL-topology we have $\mathrm{lk}(x, X) = \partial\sigma * \mathrm{lk}(\sigma, X)$ and $\mathrm{st}(x, X) = \sigma * \mathrm{lk}(\sigma, X)$. Therefore

$$
\begin{aligned}
& \mathrm{st}(x, X) - \mathrm{lk}(x, X) - \sigma \\
= \;& \sigma \cup \sigma \times \mathrm{lk}(\sigma, X) \times [0, 1] \cup \mathrm{lk}(\sigma, X) - \partial\sigma \cup \partial\sigma \times \mathrm{lk}(\sigma, X) \times [0, 1] \cup \mathrm{lk}(\sigma, X) - \sigma \\
= \;& \dot{\sigma} \times \mathrm{lk}(\sigma, X) \times (0, 1) \\
\cong \;& \mathbf{R}^{\dim \sigma + 1} \times \mathrm{lk}(\sigma, X).
\end{aligned}
$$

Since this is an open subset of a boundaryless Eulerian manifold $X - Y$, we see that $\mathbf{R}^{\dim \sigma + 1} \times \mathrm{lk}(\sigma, X)$ is a boundaryless Eulerian manifold of dimension of the same parity as $\dim X$. By repeatedly applying Proposition 2.6, we see that $\mathrm{lk}(\sigma, X)$ is a boundaryless Eulerian manifold of dimension of the same parity as $\dim X - \dim \sigma - 1$.

In case $\sigma \subset Y$, we should additionally consider $\mathrm{lk}(\sigma, Y)$. The same argument shows that $\mathbf{R}^{\dim \sigma + 1} \times (\mathrm{lk}(\sigma, X), \mathrm{lk}(\sigma, Y))$ is an Eulerian 2-strata space with relative Euler characteristic β. Therefore by Proposition 2.6, $(\mathrm{lk}(\sigma, X), \mathrm{lk}(\sigma, Y))$ is an Eulerian 2-strata space with relative Euler characteristic $(-1)^{\dim \sigma + 1}\beta$.

The argument above has only one defect. If σ is a vertex x, then there is no interior point. We may instead directly consider $\mathrm{st}(x, X) - \mathrm{lk}(x, X) - x \cong \mathbf{R} \times \mathrm{lk}(x, X)$ and obtain the same results. In fact, this is also the proof of Lemma 2.8.

$\square$

Proposition 2.10 *The lower stratum of an Eulerian 2-strata space is an Eulerian manifold without boundary. Moreover, the dimension of the lower stratum has different parity from the dimension of the whole space.*

Proof: Let (X^n, Y) be an Eulerian 2-strata space with relative Euler characteristic β. Let $x \in Y$. By Lemma 2.8, $(\mathrm{lk}(x, X), \mathrm{lk}(x, Y))$ is also an Eulerian 2-strata space with relative Euler characteristic $-\beta$, and its dimension has different parity from $n = \dim X$. Therefore by Corollary 2.4,

$$(2 + (-1)^n \beta)\chi(\mathrm{lk}(x, Y)) = (1 - (-1)^{n-1})\chi(\mathrm{lk}(x, X)).$$

On the other hand, by the definition of the relative Euler characteristic

$$\chi(\mathrm{lk}(x, X)) = 1 + (-1)^n + \beta.$$

Thus we have

$$\begin{aligned} (2 + (-1)^n \beta)\chi(\mathrm{lk}(x, Y)) &= (1 - (-1)^{n-1})(1 + (-1)^n + \beta) \\ &= (1 - (-1)^{n-1})(2 + (-1)^n \beta). \end{aligned}$$

Since $\beta \neq (-1)^{n-1}2$, we conclude that $\chi(\mathrm{lk}(x, Y)) = 1 - (-1)^{n-1}$. This shows that Y is an Eulerian manifold without boundary, and its dimension has the same parity as $n - 1$.

$\square$

Proposition 2.11 *Suppose* (X_1, Y) *and* (X_2, Y) *are Eulerian 2-strata spaces with relative Euler characteristic* β_1 *and* β_2 *respectively. Then* $(X_1 \cup_Y X_2, Y)$ *is an Eulerian 2-strata space with relative Euler characteristic* $\beta_1 + \beta_2$*.*

Note that if $\beta_1 + \beta_2 = (-1)^{n-1}2$, then the conclusion really means that $X_1 \cup_Y X_2$ is an Eulerian manifold without boundary.

If $Y = \emptyset$, then $X_1 \sqcup X_2$ is an Eulerian manifold without boundary if and only if $\dim X_1$ and $\dim X_2$ have the same parity. Otherwise $X_1 \sqcup X_2$ is an Eulerian 2-strata space with strata X_1 and X_2.

Proof of Proposition 2.11: By Proposition 2.10, Y is an Eulerian manifold without boundary. Let $\dim Y = n - 1$. Then $\dim X_1$ and $\dim X_2$ have the same parity as n. Thus for $x \in X_i - Y$ we have

$$\chi(\mathrm{lk}(x, X_1 \cup_Y X_2)) = \chi(\mathrm{lk}(x, X_i)) = 1 - (-1)^n,$$

and for $x \in Y$ we have

$$\begin{aligned}
\chi(\mathrm{lk}(x, X_1 \cup_Y X_2)) &= \chi(\mathrm{lk}(x, X_1) \cup_{\mathrm{lk}(x,Y)} \mathrm{lk}(x, X_2)) \\
&= \chi(\mathrm{lk}(x, X_1)) + \chi(\mathrm{lk}(x, X_2)) - \chi(\mathrm{lk}(x, Y)) \\
&= 1 + (-1)^n + \beta_1 + 1 + (-1)^n + \beta_2 - (1 - (-1)^{n-1}) \\
&= 1 + (-1)^n + \beta_1 + \beta_2.
\end{aligned}$$

$\square$

2.4 Topological Structure of Eulerian 2-Strata Spaces

The definition of Eulerian 2-strata spaces is motivated from topological 2-strata spaces. The main result of this section is that Eulerian 2-strata spaces has similar description as topological 2-strata spaces, with homeomorphisms (between different fibers in a bundle) replaced by having the same Euler characteristic.

Theorem 2.12 *A polyhedron pair (X^n, Y) is an Eulerian 2-strata space with relative Euler characteristic β if and only if*

1. *$X - Y$ and Y are Eulerian manifolds;*

2. *a neighborhood of Y in X is the mapping cylinder of a PL-map $f : L \to Y$ such that for any $y \in Y$, $\chi(f^{-1}(y)) = (-1)^{n-1}\beta$.*

Proof: From the theory of PL-topology, we know that Y has a regular neighborhood N in X, which is unique up to ambient isotopy of X relative to Y. Moreover, there is a closed subpolyhedron $L \subset N$ (the boundary of N) and a PL-map $f : L \to Y$, such that N is PL-homeomorphic to the mapping cylinder $M(f)$, and the homeomorphism is identity from the subpolyhedron $Y \subset N$ to the base $Y \subset M(f)$.

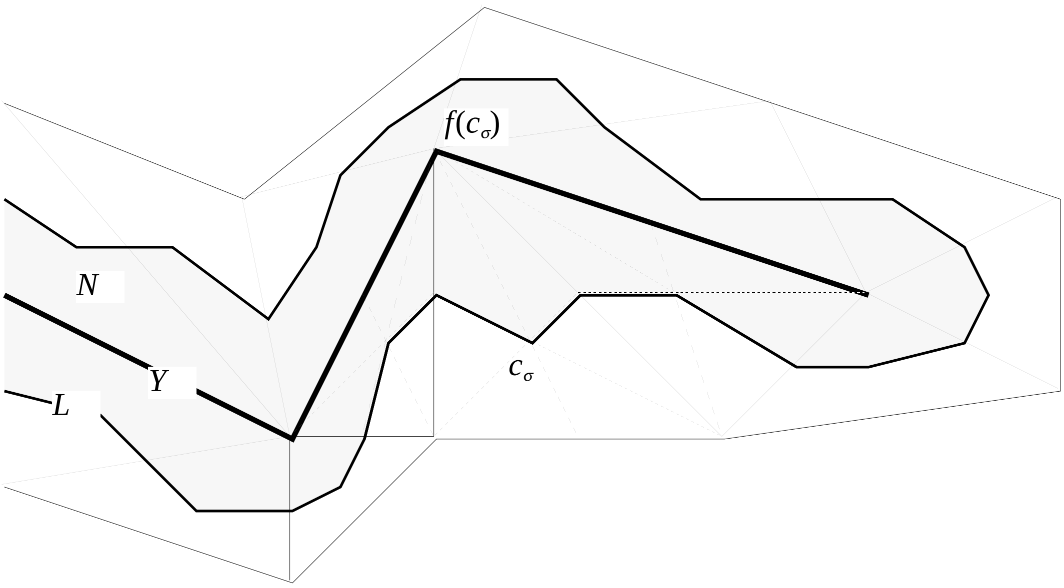

Figure 2: mapping cylinder neighborhood of lower stratum

For those who are not familiar with PL-topology, here is the explicit construction of N, L and f. Let Δ be a triangulation of (X, Y). We say Δ is *full* with respect to Y if any simplex with all vertices lying in Y is contained in Y. Since the first barycentric subdivision of any triangulation is always full, we may assume that Δ is already full. Let Δ' be the first barycentric subdivision of Δ. Then we may take

$$N=\bigcup_{\rho\in\Delta',\rho\cap Y\neq\emptyset}\rho,\qquad L=\bigcup_{\tau\in\Delta',\tau\subset N,\tau\cap Y=\emptyset}\tau \tag{6}$$

to be the simplicial neighborhood and the simplicial link of Y in Δ'. The vertices of Δ' are the centers c_σ of simplices σ of Δ. We make the following

Claim: For each $\sigma\in\Delta$, the center c_σ of σ is a vertex of N if and only if $\sigma\cap Y\neq\emptyset$.

In fact, if c_σ is a vertex in N, then it is a vertex of some $\rho\in\Delta'$ with $\rho\cap Y\neq\emptyset$. As a simplex of Δ', the vertices of ρ are $c_{\sigma_0},c_{\sigma_1},\cdots,c_{\sigma_k}$ for some simplices $\sigma_0\subset\sigma_1\subset\cdots\subset\sigma_k$ of Δ. Then $\rho\cap Y\neq\emptyset$ implies that $c_{\sigma_j}\in Y$ for some j, i.e., $\sigma_j\subset Y$. On the other hand, since c_σ is a vertex of ρ, we have $c_\sigma=c_{\sigma_i}$ for some i, i.e., $\sigma=\sigma_i$. If $i\leq j$, then $\sigma=\sigma_i\subset\sigma_j\subset Y$, so that $\sigma\cap Y=\sigma\neq\emptyset$. If $i\geq j$, then $\sigma=\sigma_i\supset\sigma_j\subset Y$, so that $\sigma\cap Y\supset\sigma_j\neq\emptyset$. For the converse, we only need to observe that if c_σ is not already lying in Y, then the segment $[c_{\sigma\cap Y},c_\sigma]$ is a 1-simplex in N (note that we need the fullness of Δ in order for $\sigma\cap Y$ to be a simplex).

The discussion also shows that a vertex c_σ of N is in L if and only if $\sigma\neq\sigma\cap Y$. The map $f:L\to Y$ is then given by

$$f(c_\sigma)=c_{\sigma\cap Y}$$

over vertices.

Now the proof of the theorem is reduced to the following lemma.

Lemma 2.13 *Suppose $f:L\to Y$ is a PL-map. Then the open mapping cylinder*

$$\dot{M}(f)=\frac{L\times[0,1)\sqcup Y}{(z,0)\sim f(z)}$$

is an Eulerian 2-strata space with relative Euler characteristic β and with Y as the lower stratum if and only if

1. *Y is an Eulerian manifold without boundary, and $\dim L-\dim Y$ is even;*

2. *for any $y\in Y$, $\chi(f^{-1}(y))=(-1)^{\dim Y}\beta$.*

Proof: If $(\dot{M}(f), Y)$ is an Eulerian 2-strata space, then Propositions 2.10 implies the necessity of the first condition. Therefore we need only to show that under the first condition, $(\dot{M}(f), Y)$ is an Eulerian 2-strata space if and only if the second condition is satisfied.

We fix triangulations Δ_L on L and Δ_Y on Y so that f is a simplicial map. Then for any fixed simplex $\sigma \subset Y$, $f^{-1}(y)$ are homeomorphic for all $y \in \dot{\sigma}$. In fact, $f^{-1}(\dot{\sigma})$ is homeomorphic to $\dot{\sigma} \times f^{-1}(y)$, so that

$$\chi(f^{-1}(\dot{\sigma})) = (-1)^{\dim \sigma}\chi(f^{-1}(y)). \tag{7}$$

For each simplex σ, fix an interior point y_σ of σ. Then $\mathrm{st}(y_\sigma, Y) = \sigma * \mathrm{lk}(\sigma, Y)$ is the geometric realization of the subcomplex

$$\{\rho : \ \rho \subset \tau \supset \sigma \text{ for some } \tau \in \Delta_Y\},$$

and $\mathrm{lk}(y_\sigma, Y) = \partial\sigma * \mathrm{lk}(\sigma, Y)$ is the geometric realization of the subcomplex

$$\{\rho : \ \rho \subset \tau \supset \sigma \text{ for some } \tau \in \Delta_Y, \ \sigma \ \not\subset \rho\}.$$

So we have

$$\dot{\mathrm{st}}(y_\sigma, Y) = \mathrm{st}(y_\sigma, Y) - \mathrm{lk}(y_\sigma, Y) = \bigsqcup_{\sigma \subset \rho \in \Delta_Y} \dot{\rho}, \tag{8}$$

and

$$f^{-1}(\dot{\mathrm{st}}(y_\sigma, Y)) = \bigsqcup_{\sigma \subset \rho \in \Delta_Y} f^{-1}(\dot{\rho}). \tag{9}$$

We define functions ϕ and ψ on the simplices of Δ_Y:

$$\phi(\sigma) = \chi(f^{-1}(y_\sigma)),$$

$$\psi(\sigma) = \chi(f^{-1}(\dot{\mathrm{st}}(y_\sigma, Y))) \overset{(9)}{=} \sum_{\sigma \subset \rho \in \Delta_Y} \chi(f^{-1}(\dot{\rho})) \overset{(7)}{=} \sum_{\sigma \subset \rho \in \Delta_Y} (-1)^{\dim \rho}\phi(\rho). \tag{10}$$

Since Y is an Eulerian manifold without boundary, $\chi(\mathrm{lk}(y_\sigma, Y)) = 1 - (-1)^{\dim Y}$. Then $\chi(\dot{\mathrm{st}}(y_\sigma, Y)) = \chi(\mathrm{st}(y_\sigma, Y)) - \chi(\mathrm{lk}(y_\sigma, Y)) = (-1)^{\dim Y}$. If ϕ is constant, then

$$\psi(\sigma) \overset{(10)}{=} \left(\sum_{\sigma \subset \rho \in \Delta_Y} (-1)^{\dim \rho}\right)\phi \overset{(8)}{=} \chi(\dot{\mathrm{st}}(y_\sigma, Y))\phi = (-1)^{\dim Y}\phi \tag{11}$$

is independent of σ. Conversely, by the Möbius inversion, (10) is equivalent to

$$\phi(\sigma) = \sum_{\sigma \subset \rho \in \Delta_Y} (-1)^{\dim \rho} \psi(\rho).$$

By the same argument, if ψ is constant, then ϕ is also constant.

For the open mapping cylinder $\dot{M}(f)$, it can be seen from PL-topology that

$$\mathrm{lk}(y_\sigma, \dot{M}(f)) \;\; \cong \;\; f^{-1}(\dot{\mathrm{st}}(y_\sigma, Y)) \sqcup M(f_{y_\sigma}), \tag{12}$$

where $M(f_{y_\sigma})$ is the mapping cylinder of $f_{y_\sigma} : f^{-1}(\mathrm{lk}(y_\sigma, Y)) \longrightarrow \mathrm{lk}(y_\sigma, Y)$. Since $M(f_{y_\sigma})$ is homotopic to $\mathrm{lk}(y_\sigma, Y)$, $\chi(M(f_{y_\sigma})) = \chi(\mathrm{lk}(y_\sigma, Y))$. Counting the Euler characteristic of both sides of (12), we have

$$\begin{aligned} \chi(\mathrm{lk}(y_\sigma, \dot{M}(f)) &= \chi(f^{-1}(\dot{\mathrm{st}}(y_\sigma, Y))) + \chi(\mathrm{lk}(y_\sigma, Y)) \\ &= \psi(\sigma) + 1 - (-1)^{\dim Y}. \end{aligned}$$

For any $y \in \dot{\sigma}$, $\mathrm{lk}(y, \dot{M}(f))$ is homeomorphic to $\mathrm{lk}(y_\sigma, \dot{M}(f))$. Thus

$$\chi(\mathrm{lk}(y, \dot{M}(f))) = \psi(\sigma) + 1 - (-1)^{\dim Y}. \tag{13}$$

The left side is independent of y if and only if ψ is independent of σ. However, ψ is independent of σ if and only if ϕ is independent of σ. Moreover, since $f^{-1}(y)$ is homeomorphic to $f^{-1}(y_\sigma)$ for $y \in \dot{\sigma}$, $\chi(f^{-1}(y)) = \chi(f^{-1}(y_\sigma))$, so that ϕ is independent of σ if and only $\chi(f^{-1}(y))$ is independent of y. Thus we have shown that $\chi(\mathrm{lk}(y, \dot{M}(f))$ is constant if and only if $\chi(f^{-1}(y))$ is constant.

Numerically, whenever $\chi(f^{-1}(y))$ or $\chi(\mathrm{lk}(y, \dot{M}(f))$ is constant, (11) and (13) imply

$$\chi(\mathrm{lk}(y, \dot{M}(f)) = 1 - (-1)^{\dim Y} + (-1)^{\dim Y} \chi(f^{-1}(y)). \tag{14}$$

Since $\dim L - \dim Y$ is assumed even, the left is equal to $1 + (-1)^{\dim L + 1} + \beta$ if and only if $\chi(f^{-1}(y)) = (-1)^{\dim Y} \beta$.

$\square$

2.5 2-Dimensional Eulerian 2-Strata Spaces

One-dimensional Eulerian 2-strata spaces are regular graphs (with loops and multiple edges allowed), i.e., the graphs such that every vertex is incident with equal number of edges (loops counted twice).

Before we study 2-dimensional Eulerian 2-strata spaces, we make the following general remark: Suppose X is an n-dimensional polyhedron. We call a point $x \in X$ an *n-dimensional point* if a neighborhood of x in X is homeomorphic to $\mathbf{R}^n$. These points form an n-dimensional manifold and an open subset of X. Given any triangulation of X, the set of n-dimensional points of X is the union of the interior of some simplices, including all n-dimensional simplices. Therefore the complement of n-dimensional points is a closed subcomplex of dimension $\leq n-1$.

Let (X, Y) be a 2-dimensional Eulerian 2-strata space with relative Euler characteristic $\beta \neq -2$. The lower stratum Y must be contained in the complement Z of 2-dimensional points of X. We fix a triangulation of X. Then Y is a closed subcomplex of dimension ≤ 1.

At an isolated point $y \in Z$, the link $\mathrm{lk}(y, X)$ is, by Lemma 2.9, a 1-dimensional Eulerian manifold without boundary. Therefore it is a disjoint union of circles. In particular, $\chi(\mathrm{lk}(y, Y)) = 0$ and $y \in X - Y$.

Next we consider a 1-simplex in Z. Suppose it is the face of l 2-simplices. Then for any point y in the interior of the 1-simplex, $\mathrm{lk}(y, X)$ is 2 points connected by l arcs, so that $\chi(\mathrm{lk}(y, X)) = 2 - l$. Since y is not a 2-dimensional point, we have $l \neq 2$. Thus the interior of the 1-simplex belongs to Y, and $\beta = -l$. In particular, l is independent of the choice of 1-simplices in Z.

Finally we consider a non-isolated 0-simplex $y \in Z$. The link $\mathrm{lk}(y, X)$ is a graph (1-dimensional polyhedron, with loops and multiple edges allowed). The vertices of the graph correspond to 1-simplices of Z with y as an end point. The edges of the graph correspond to the 2-simplices containing the vertex y. Since each 1-simplex of Z is incident with $-\beta$ 2-simplices, each vertex of the graph is incident with $-\beta$ edges (a loop at a vertex is counted twice). Let V and E be the numbers of vertices and edges in the graph. Then we have $2E = -\beta V$, and

$$\chi(\mathrm{lk}(y, X)) = V - E = \frac{2+\beta}{2} V.$$

Since $\beta \neq -2$ and $V \neq 0$ (since y is not isolated), we see that $\chi(\mathrm{lk}(y, X)) \neq 0$, so that $y \in Y$. Therefore $\chi(\mathrm{lk}(y, X)) = 2 + \beta$. Again by $\beta \neq -2$ we conclude that $V = 2$.

Note that $V = 2$ means y is the end point of exactly two 1-simplices of Z. As a result, Z is a disjoint union of isolated points and circles. However, on the circles we have special marks (the non-isolated 0-simplices in Z) where the local structure of X is somewhat complicated. Away from these marks we simply have $-\beta$ branches of surfaces meeting together.

Coming back to these marks, the graph $\mathrm{lk}(y, Z)$ consists of 2 vertices and $-\beta$ edges. Moreover, each vertex is incident with $-\beta$ edges (loops counted twice). In such a graph, we have i and j, such that

(1) $i + 2j = -\beta$,

(2) the two vertices are connected by i edges (called connecting edges),

(3) each of the two vertices are attached by j loops.

Taking the cone of the graph, which is a neighborhood of y in X, we see that each connecting edge becomes a half plane with the arc around the marked point as the boundary. Moreover, each loop becomes a cone with marked point as the cone point, and half of the arc as a cone line.

If $j = 0$ at the marked point, then we simply get $-\beta$ branches of surfaces meeting along the arc near the point. At these points the branches of surfaces may be permuted. In case the permutations are trivial, the marks can be removed.

If $j \neq 0$ at the marked point, then we have j foldings on both sides of the marked point. These marks cannot be removed.

If we delete an open mapping cylinder neighborhood of Z from X, then we get a 2-dimensional manifold S with boundary. The space X can be reconstructed by gluing the boundary of S to Z. The gluing scheme is given by the discussion above.

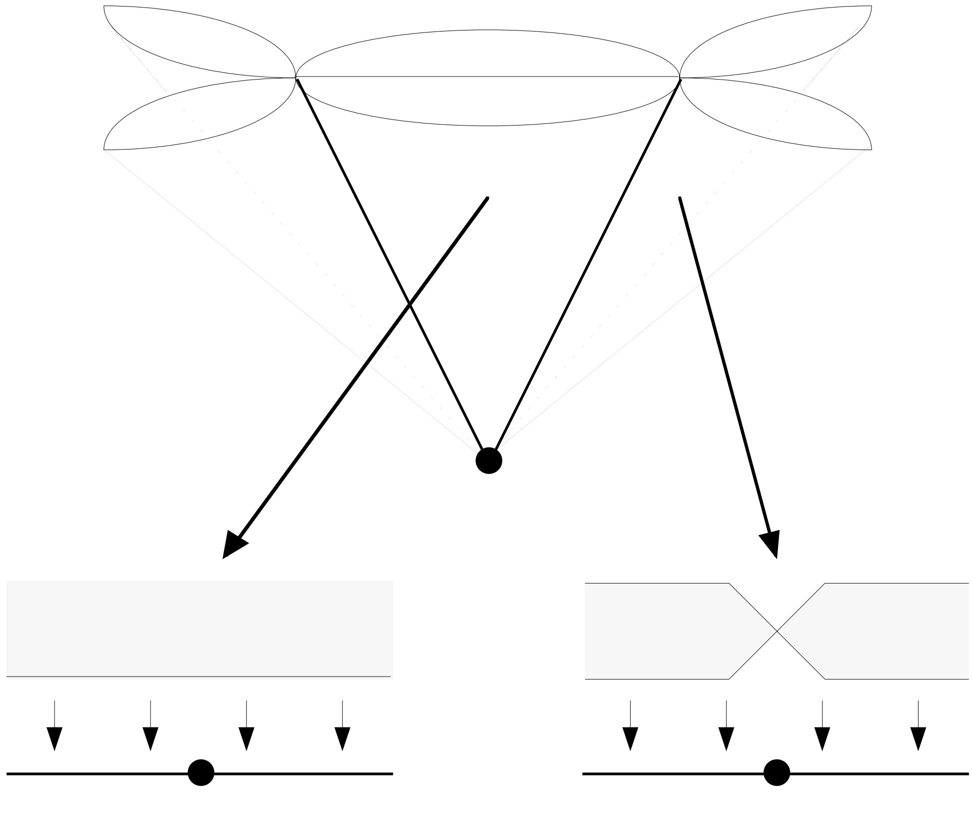

Figure 3: cone on the link of a marked point on a circle and the glueing of branches

The gluing map $f : \partial S \to Z$ sends some circles in ∂S to isolated points in Z and sends some other circles to Y =union of circles in Z. The map f is mostly locally homeomorphic between circles, except some "symmetric" foldings at finitely many marked points. In particular, the inverse image $f^{-1}(y)$ always consists of $-\beta$ points for $y \in Y$.

We summarize the discussion above in the following theorem.

Theorem 2.14 *Any compact 2-dimensional 2-strata space with relative Euler characteristic β is given by a surface S, a disjoint union Y of circles, and a map $f : \partial S \to Y \cup \{\text{points}\}$, satisfying*

1. *away from finitely many points of Y, f is a local homeomorphism;*
2. *for any point $y \in Y$, $f^{-1}(y)$ consists of $-\beta$ points.*

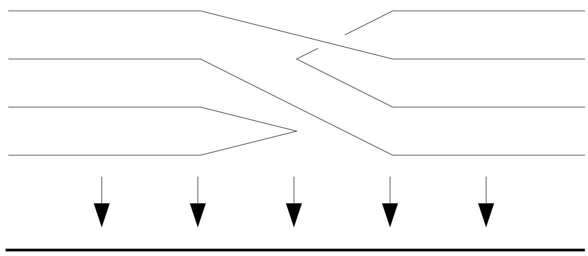

Figure 4: map between arcs such that each point has 4 preimages

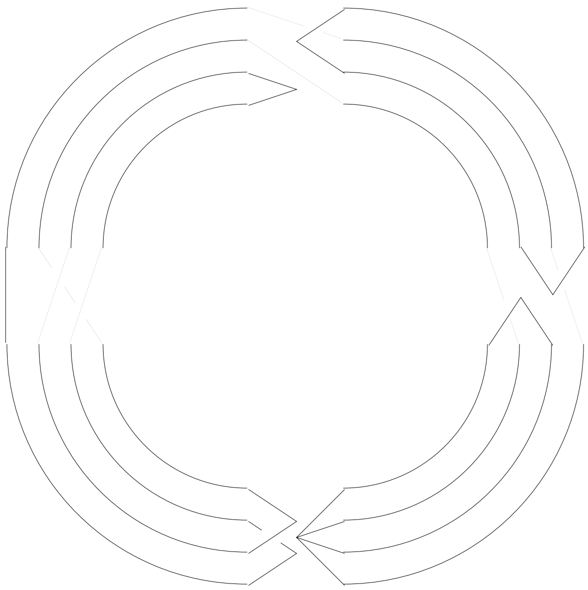

Figure 5: map between circles such that each point has 4 preimages

3 Linear Conditions on f-Vectors

3.1 Relations Between f-Vectors

Let (X^n, Y) be a compact Eulerian 2-strata space. For any triangulation Δ of (X, Y), there are three relevant f-vectors:

$f(X;\Delta)$	for whole X
$f(Y;\Delta)$	for lower stratum Y
$f(X,Y;\Delta) = f(X;\Delta) - f(Y;\Delta)$	for upper stratum $X - Y$.

The following theorem indicates all the relations between these vectors.

Theorem 3.1 *Let (X^n, Y) be a compact Eulerian 2-strata space with relative Euler characteristic β. Then*

$$D(n)f(X;\Delta) = (2+(-1)^n\beta)f(Y;\Delta) \tag{15}$$

$$D(n)f(X,Y;\Delta) = (-1)^n\beta f(Y;\Delta) \tag{16}$$

$$(2+(-1)^n\beta - D(n))f(X;\Delta) = (2+(-1)^n\beta)f(X,Y;\Delta) \tag{17}$$

$$((-1)^n\beta + D(n))f(X,Y;\Delta) = (-1)^n\beta f(X;\Delta) \tag{18}$$

Proof: (15) is the generalized Dehn-Sommerville equation in Theorem 2.3. (17) is a consequence of (15):

$$\begin{aligned}(2+(-1)^n\beta - D(n))f(X;\Delta) &= (2+(-1)^n\beta)f(X;\Delta) - D(n)f(X;\Delta)\\ &= (2+(-1)^n\beta)f(X;\Delta) - (2+(-1)^n\beta)f(Y;\Delta)\\ &= (2+(-1)^n\beta)f(X,Y;\Delta).\end{aligned}$$

By Proposition 2.11, the double $(X\cup_Y X, Y)$ is an Eulerian 2-strata space with relative Euler characteristic 2β. Therefore by (15),

$$\begin{aligned}(2+(-1)^n 2\beta)f(Y;\Delta) &= D(n)f(X\cup_Y X;\Delta\cup_{\Delta|_Y}\Delta)\\ &= D(n)(f(X;\Delta)+f(X,Y;\Delta))\\ &= (2+(-1)^n\beta)f(Y;\Delta) + D(n)f(X,Y;\Delta).\end{aligned}$$

This proves (16). (18) may be then obtained as a consequence of (16):

$$\begin{aligned}(-1)^n\beta f(X;\Delta) = &\ (-1)^n\beta f(X,Y;\Delta)+(-1)^n\beta f(Y;\Delta)\\ = &\ (-1)^n\beta f(X,Y;\Delta)+D(n)f(X,Y;\Delta)\\ = &\ ((-1)^n\beta+D(n))f(X,Y;\Delta).\end{aligned}$$

□

Because $\beta \neq (-1)^{n-1}2$, (15) and (17) imply that $f(Y;\Delta)$ and $f(X,Y;\Delta)$ are rationally determined by $f(X;\Delta)$. If $\beta \neq 0$, then (16) and (18) imply that $f(Y;\Delta)$ and $f(X;\Delta)$ are also rationally determined by $f(X,Y;\Delta)$.

Theorem 3.1 generalizes our results on f-vectors of manifolds with boundary in [12]. It also indicates that the linear conditions on $f(X;\Delta)$ and $f(X,Y;\Delta)$ will not be as simple as in [12]. The answer should depend on whether β is zero or not, and some torsion conditions will appear.

3.2 Integral Linear Conditions

Denote

$$f(X)=\{f(X;\Delta):\Delta \text{ is a triangulation of } X\},$$

$$f(X,Y)=\{f(X,Y;\Delta):\Delta \text{ is a triangulation of } X\}.$$

If T is a triangulation of Y that is extendable to a triangulation of the whole X, then we use $f(X;\mathrm{rel}\ T)$ and $f(X,Y;\mathrm{rel}\ T)$ to denote the similar collections of f-vectors of all triangulations that restrict to T on the boundary.

The following characterization of the integral affine span of f-vectors of triangulations with a prescribed boundary triangulation is also similar to Theorem 2 of [12]. The proof is similar and therefore is omitted here.

Theorem 3.2 *Suppose (X^n,Y) is a compact Eulerian 2-strata space with relative Euler characteristic β. Then the integral affine span of $f(X;\mathrm{rel}\ T)$ is characterized by*

$$\begin{cases}\chi(v) &= \chi(X)\\ D(n)v &= (2+(-1)^n\beta)f(Y;T)\end{cases}\tag{19}$$

and the integral affine span of $f(X,Y;\mathrm{rel}\,T)$ is characterized by

$$\left\{\begin{array}{lcl}\chi(v) & = & (-1)^n\chi(X)\\ D(n)v & = & (-1)^n\beta f(Y;T)\end{array}\right. \tag{20}$$

In particular, the rank of the affine span of $f(X;\mathrm{rel}\,T)$ or $f(X,Y;\mathrm{rel}\,T)$ is $\lfloor\frac{n+1}{2}\rfloor$.

To characterize the integral affine span of f-vectors of triangulations with no restriction on the lower stratum, we need to introduce some notations. From the explicit expression of the Dehn-Sommerville matrix $D(n):\mathbf{Z}^{n+1}\to\mathbf{Z}^n$ in [12], we see that if $m<n$ and have the same parity, then

$$D(n)=\left(\begin{array}{cc}D(m) & F(m,n)\\ 0 & E(m,n)\end{array}\right)$$

for some matrices

$$\begin{array}{rl}E(m,n): & \mathbf{Z}^{n-m}\to\mathbf{Z}^{n-m},\\ F(m,n): & \mathbf{Z}^{n-m}\to\mathbf{Z}^{m}.\end{array}$$

The identity $D(n-1)D(n)=0$ (see Corollary 6 of [12]) implies

$$D(m-1)F(m,n)=-F(m-1,n-1)E(m,n). \tag{21}$$

The identity (4) implies

$$\chi F(m,n)+(-1)^m\chi E(m,n)=(1-(-1)^n)(-1)^{m+1}\chi. \tag{22}$$

Theorem 3.3 *Suppose (X^n,Y) is a compact Eulerian 2-strata space with relative Euler characteristic β. If $Y\neq\emptyset$ and $\dim Y=r$, then*

1. *the integral affine span of $f(X)$ is characterized by*

$$\left\{\begin{array}{lcl}\chi(v) & = & \chi(X)\\ (D(r+1),F(r+1,n))v & = & 0 \bmod (2+(-1)^n\beta)\\ (0,E(r+1,n))v & = & 0\end{array}\right. \tag{23}$$

2. the integral affine span of $f(X,Y)$ is characterized by

$$\begin{cases} \chi(v) & = & (-1)^n \chi(X) \\ (D(r+1), F(r+1,n))v & = & 0 \bmod \beta \\ (0, E(r+1,n))v & = & 0 \end{cases} \tag{24}$$

for $\beta \neq 0$, and by

$$\begin{cases} \chi(v) & = & (-1)^n \chi(X) \\ D(n)v & = & 0 \end{cases} \tag{25}$$

for $\beta = 0$.

In particular, the rank of the affine span of $f(X)$ (and $f(X,Y)$ in case $\beta \neq 0$) is $\lfloor \frac{n+r+1}{2} \rfloor$.

Remark 3 By Proposition 2.10, $r+1$ and n have the same parity. Thus the decomposition of $D(n)$ used in the theorem makes sense.

As an example, consider k copies ($k \geq 2$) of the sphere S^4 with the circle S^1 embedded inside. By gluing these together along the circle we obtain a 2-strata space (X^4, S^1) with relative Euler characteristic $\beta = -2k$ and $\chi(X) = 2k$. The integral linear conditions on the f-vectors of triangulations of X are

$$\begin{cases} f_0 - f_1 + f_2 - f_3 + f_4 = 2k \\ 2f_1 - 3f_2 + 4f_3 - 5f_4 = 0 \bmod (2-2k) \\ 2f_3 - 5f_4 = 0. \end{cases}$$

For $k = 2$, we see that f_3 is divisible by 5, f_2 and f_4 are divisible by 2.

Proof of Theorem 3.3: The proof is a modification of the proof of Theorem 3 in [12], with special care on the torsion part. We start by fixing a triangulation Δ of (X,Y). As in the proof of Theorem 12 in [12], we may make use of special triangulations of D^r and D^n to modify Δ and obtain triangulations (similar to (20) on page 160 of [12])

$$\Delta_1, \Delta_2, \cdots, \Delta_{\lfloor \frac{r+1}{2} \rfloor}; \quad \Delta_{\delta_0}, \Delta_{\delta_1}, \cdots, \Delta_{\delta_{\lfloor \frac{n+1}{2} \rfloor}} \tag{26}$$

of (X,Y). These triangulations have the following three properties

1. The triangulations Δ_{δ_j}, $0 \le j \le \lfloor \frac{n+1}{2} \rfloor$, restrict to the same triangulation on Y;

2. The integral linear span A'_r of $f(Y;\Delta_i) - f(Y;\Delta_{\delta_0})$, $1 \le i \le \lfloor \frac{r+1}{2} \rfloor$, is a direct summand of $\mathbf{Z}^{r+1}$ of rank $\lfloor \frac{r+1}{2} \rfloor$ (see (25) in [12]);

3. The integral linear span A'_n of $f(X;\Delta_{\delta_j}) - f(X;\Delta_{\delta_0})$, $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$, is a direct summand of $\mathbf{Z}^{n+1}$ of rank $\lfloor \frac{n+1}{2} \rfloor$ (see (26) in [12]).

Denote by A' the integral linear span of the following differences of f-vectors

$$
\begin{aligned}
&f(X;\Delta_i) - f(X;\Delta_{\delta_0}), \quad 1 \le i \le \lfloor \tfrac{r+1}{2} \rfloor \\
&f(X;\Delta_{\delta_j}) - f(X;\Delta_{\delta_0}), \quad 1 \le j \le \lfloor \tfrac{n+1}{2} \rfloor
\end{aligned}
$$

Denote

$$\alpha = 2 + (-1)^n \beta$$

and the composition

$$\bar{D}(n) : \mathbf{Z}^{n+1} \xrightarrow{D(n)} \mathbf{Z}^n \xrightarrow{\mathrm{mod}\alpha} \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}.$$

Then we claim

$$A' \subset \text{integral linear span}\{f(X) - f(X;\Delta_{\delta_0})\} \subset \ker(\chi, \bar{D}(n)). \tag{27}$$

The first inclusion follows from the definition of A'. The second inclusion follows from (15) and $\dim Y = r$. The characterization (23) may be interpreted as the second and the third terms of (27) are equal. We will prove this by showing that the first and the third terms of (27) are equal.

Note that A'_n is natually included in A'. Moreover,

$$D(n)[f(X;\Delta_i) - f(X;\Delta_{\delta_0})] \overset{(15)}{=} \alpha[f(Y;\Delta_i) - f(Y;\Delta_{\delta_0})], \quad 1 \le i \le \lfloor \frac{r+1}{2} \rfloor, \tag{28}$$

and

$$D(n)[f(X;\Delta_{\delta_j}) - f(X;\Delta_{\delta_0})] \overset{(15)}{=} \alpha[f(Y;\Delta_{\delta_j}) - f(Y;\Delta_{\delta_0})] = 0, \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor, \tag{29}$$

where the second equality comes from the first property of (26). Then (28) and (29) imply that $\alpha^{-1}D(n)$ restricts to a map $A' \to A'_r$. On the other hand, the second property of (26) implies that (28) is linearly independent. Therefore the kernel of $\alpha^{-1}D(n) : A' \to A'_r$ is A'_n. Thus we are able to construct the following diagram where the bar means appropriate reduction of certain matrix entries modulo α, and $\bar{\Phi}(r, n-1)$ is the reduction of

$$\Phi(r, n-1) = \begin{pmatrix} 1-(-1)^n & 0 & (-1)^r\chi \\ 0 & I & 0 \\ 0 & 0 & -F(r, n-1) \end{pmatrix}.$$

The diagram is commutative. The only nontrivial commutativity is the lower right square. Since

$$\begin{pmatrix} \chi \\ D(n) \end{pmatrix} = \begin{pmatrix} \chi & (-1)^r\chi \\ D(r+1) & F(r+1, n) \\ 0 & E(r+1, n) \end{pmatrix},$$

we have

$$\Phi(r, n-1) \begin{pmatrix} \chi \\ D(n) \end{pmatrix} = \begin{pmatrix} (1-(-1)^n)\chi & (1-(-1)^n)(-1)^r\chi + (-1)^r\chi E(r+1, n) \\ D(r+1) & F(r+1, n) \\ 0 & -F(r, n-1)E(r+1, n) \end{pmatrix}, \tag{30}$$

and

$$\begin{pmatrix} \chi \\ I \\ D(r) \end{pmatrix} (D(r+1), F(r+1, n)) = \begin{pmatrix} \chi D(r+1) & \chi F(r+1, n) \\ D(r+1) & F(r+1, n) \\ D(r)D(r+1) & D(r)F(r+1, n) \end{pmatrix}. \tag{31}$$

By $D(r)D(r+1) = 0$, $\chi D(r+1) = (1-(-1)^{r+1})\chi$, and taking $m = r+1$ in (21) and (22), we see that (30) and (31) are equal.

We now show that the middle row of the diagram is exact. We have already argued that the left column is exact. The proof of Theorems 2 and 3 in [12] (see page 164) implies that $A'_n = \ker(\chi, D(n))$ and $A'_r = \ker(\chi, D(r))$. The equality $A'_n = \ker(\chi, D(n))$ means that the top row of the diagram is exact. The equality $A'_r = \ker(\chi, D(r))$ easily implies that the bottom row is exact. Finally, the inclusion (27) implies that the composition of the maps at the middle row of the diagram is 0.

Suppose $x \in \mathbf{Z}^{n+1}$ satisfies $(\chi, \bar{D}(n))x = 0$. Then we have

$$(\chi, \bar{I}, D(r))(D(r+1), F(r+1, n))x = \bar{\Phi}(r, n-1)(\chi, \bar{D}(n))x = 0.$$

Therefore by the exactness of the bottom row, $(D(r+1), F(r+1, n))x = \alpha y$ for some $y \in A'_r$. By the exactness of the left column, $y = \alpha^{-1}(D(r+1), F(r+1, n))z$ for some $z \in A'$. Thus

$$(D(r+1), F(r+1, n))x = (D(r+1), F(r+1, n))z. \tag{32}$$

On the other hand, $(\chi, \bar{D}(n))x = 0$ implies that

$$\chi(x) = 0, \quad (0, E(r+1, n))x = 0. \tag{33}$$

Since the composition of the maps in the middle row is 0, we also have

$$\chi(z) = 0, \quad (0, E(r+1, n))z = 0. \tag{34}$$

Putting (32), (33), and (34) together, we see that $x - z \in \ker(\chi, D(n))$. By the exactness of the top row, we obtain $x - z \in A'_n$. It then follows from $z \in A'$ that $x = (x - z) + z \in A'$.

We just proved that the right side of (27) is contained in the left side. Consequently, the inclusions in (27) are equalities. The completes the proof of the first part of Theorem 3.3.

The proof of the second part is exactly the same. For $\beta \neq 0$, all we need to do is to replace $\alpha = 2 + (-1)^n \beta$ by β. For $\beta = 0$, we may simply forget about the torsion part.

□

3.3 Linear Conditions over any Abelian Group

The linear equations (19), (20), (23), (24), and (25) determine the integral span of various collections of f-vectors. Consequently, they determine in principle all the other linear equations for f-vectors. However, there is still the problem of whether any linear condition is a linear combination of the equations in (19), (20), (23), (24), and (25).

Set $\alpha = 2 + (-1)^n\beta$ again. Take (23) for example. The integral characterization of the affine span of $f(X)$ may be interpreted as the exact sequence

$$0 \to \mathbf{Z}\text{-span}[f(X) - f(X;\Delta_0)] \xrightarrow{\text{incl}} \mathbf{Z}^{n+1} \xrightarrow{(\chi,\bar{D}(n))} \mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}, \tag{35}$$

where Δ_0 is a fixed triangulation of (X, Y). In general, given an abelian group G (such as $\mathbf{Z}/k\mathbf{Z}$), a linear condition in G over $f(X)$ is a homomorphism $\lambda : \mathbf{Z}^{n+1} \to G$ and a constant $c \in G$ such that $\lambda(f(X;\Delta)) = c$ for any triangulation Δ of (X, Y). This implies that $\lambda = 0$ on $\mathbf{Z}$-span$[f(X) - f(X;\Delta_0)]$, so that λ factors through $\text{im}(\chi, \bar{D}(n))$. If $\text{im}(\chi, \bar{D}(n))$ is a direct summand of $\mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}$, then the factorization can be extended to a homomorphism $\mu : \mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1} \to G$, and we have $\lambda = \mu(\chi, \bar{D}(n))$. This means exactly that $\lambda(v) = c$ is a linear combination of the equations in (23). In other words, (23) generates all the other linear conditions on $f(X)$.

Theorem 3.4 *Let (X^n, Y) be a compact Eulerian 2-strata space with relative Euler characteristic β. Then any linear equation over an abelian group satisfied by $f(X)$ (respectively, $f(X, Y)$, $f(X; \text{rel } T)$, $f(X, Y; \text{rel } T)$) is a linear combination of reductions of equations in* (23) (*respectively,* (24) *or* (25) *according to whether $\beta \neq 0$ or $\beta = 0$,* (19), (20)).

Proof: As discussed above, the key is to show the image of relevant equations is a direct summand. We provide the proof for the case of $f(X)$. The other cases are similar.

Specifically, we need to show that $\text{im}(\chi, \bar{D}(n)) \subset \mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}$ is a direct summand. We already know from [12] that integrally, $\text{im}(\chi, D(n)) \subset \mathbf{Z}^{n+1}$ is a direct summand. The problem is to find a specific integral projection onto the image so that it can be reduced to a projection after modulo α in some coordinates.

Let D_i^χ, $i \geq 0$, be the columns of $\binom{\chi}{D(n)}$:

$$\begin{pmatrix} \chi \\ D(n) \end{pmatrix} = (D_0^\chi, D_1^\chi, \cdots, D_n^\chi). \tag{36}$$

Then $\mathrm{im}(\chi, D(n))$ is simply the integral span of D_i^χ, $0 \le i \le n$. We note that by the explicit expression of $D(n)$ in [12], D_i^χ depends only on the parity of n, except perhaps adding zeros to the end of vectors so that the dimensions become correct. The other fact we need from the explicit expression is

$$D_0^\chi = \begin{cases} (1,0,0,0,\cdots)^t & n \text{ is even} \\ (1,2,0,0,\cdots)^t & n \text{ is odd} \end{cases} \tag{37}$$

$$D_i^\chi = \begin{cases} (*,\cdots,*,i+1,2,0,0,\cdots)^t & n-i \text{ is odd} \\ (*,\cdots,*,-\frac{1}{2}i(i+1),-(i+1),0,0,\cdots)^t & n-i \text{ is even} \end{cases} \tag{38}$$

where the last two nonzero terms are the i and $i+1$ coordinates for odd $n-i$, and are the $i-1$ and i coordinates for even $n-i$.

Let $e_i = (0,\cdots,0,1_{(i)},0,\cdots,0)^t$ be the standard basis vector (note that $e_0 = (1,0,0,\cdots)$). Consider the integral matrix

$$A(n) = (D_0^\chi, e_1, D_2^\chi + D_1^\chi, e_3, D_4^\chi + 2D_3^\chi, e_5, \cdots, e_{n-1}, D_n^\chi + \tfrac{n}{2}D_{n-1}^\chi)$$

for even n, and the integral matrix

$$A(n) = (D_0^\chi, e_1, D_2^\chi, e_3 + e_2, D_4^\chi, e_5 + 2e_4, \cdots, e_n + \tfrac{n-1}{2}e_{n-1}, D_{n-1}^\chi)$$

for odd n. It then follows from (37) and (38) that in either case, $A(n)$ is an integrally invertible matrix. In other words,

$$Z(n) = \begin{cases} \mathbf{Z}\text{-span}\{D_0^\chi, D_2^\chi + D_1^\chi, D_4^\chi + 2D_3^\chi, \cdots, D_n^\chi + \frac{n}{2}D_{n-1}^\chi\} & \text{for even } n \\ \mathbf{Z}\text{-span}\{D_0^\chi, D_2^\chi, D_4^\chi, \cdots, D_{n-1}^\chi\} & \text{for odd } n \end{cases} \tag{39}$$

and

$$Z'(n) = \begin{cases} \mathbf{Z}\text{-span}\{e_1, e_3, e_5, \cdots, e_{n-1}\} & \text{for even } n \\ \mathbf{Z}\text{-span}\{e_1, e_3 + e_2, e_5 + 2e_4, \cdots, e_n + \frac{n-1}{2}e_{n-1}\} & \text{for odd } n \end{cases} \tag{40}$$

form complementary direct summands of $\mathbf{Z}^{n+1}$. As a result, the rank of $Z(n)$ is $\left\lfloor \frac{n+2}{2} \right\rfloor$, which is the same as the rank of (36) (see [8] [16], for example). Since $Z(n) \subset \mathrm{im}(\chi, D(n))$, $\mathrm{rank} Z(n) = \mathrm{rank}\ \mathrm{im}(\chi, D(n))$, and $Z(n)$ is a direct summand, we conclude that $Z(n) = \mathrm{im}(\chi, D(n))$ (by Lemma 14 of [12], for example).

The equality $Z(n) \oplus Z'(n) = \mathbf{Z}^{n+1}$ provides an integral projection $P(n) : \mathbf{Z}^{n+1} \to \mathbf{Z}^{n+1}$ onto $Z(n) = \mathrm{im}(\chi, D(n))$ by sending $Z'(n)$ to zero. It follows from the explicit definitions (39) and (40) that if we view $\mathbf{Z}^{r+2}$ as $\mathbf{Z}^{r+2} \oplus 0 \subset \mathbf{Z}^{n+1}$, then (recall that r and n have different parity)

$$Z(n) \cap \mathbf{Z}^{r+2} = Z(r+1); \quad Z'(n) \cap \mathbf{Z}^{r+2} = Z'(r+1).$$

Consequently, we have the recursive relation

$$P(n) = \begin{pmatrix} P(r+1) & * \\ 0 & Q(r+1,n) \end{pmatrix}, \tag{41}$$

for some $Q(r+1,n) : \mathbf{Z}^{n-r-1} \to \mathbf{Z}^{n-r-1}$.

In order to show that $P(n)$ may be reduced to a projection in $\mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}$, we still need to know the 0-th row of $P(n)$. Because the 0-th coordinates of vectors in $Z'(n)$ are all 0, $P(n)$ preserves the 0-th coordinates of vectors in $Z'(n)$. On the other hand, since $P(n)$ is identity on $Z(n)$, it in particular preserves the 0-th coordinates of vectors in $Z(n)$. Consequently, $P(n)$ preserves the 0-th coordinates of all vectors. This means exactly that the 0-th row of $P(n)$ is $(1, 0, 0, 0, \cdots)$. Hence

$$P(n) = \begin{pmatrix} 1 & 0 \\ * & P'(n) \end{pmatrix}, \tag{42}$$

for some $P'(n) : \mathbf{Z}^n \to \mathbf{Z}^n$. Combining (41) and (42) together, we obtain

$$P(n) = \begin{pmatrix} 1 & 0 & 0 \\ * & P'(r+1) & * \\ 0 & 0 & Q(r+1,n) \end{pmatrix}.$$

This can be reduced to a projection

$$\bar{P}(n) = \begin{pmatrix} 1 & 0 & 0 \\ * & \bar{P}'(r+1) & * \\ 0 & 0 & Q(r+1,n) \end{pmatrix}$$

of $\mathbf{Z} \oplus \mathbf{Z}_\alpha^{r+1} \oplus \mathbf{Z}^{n-r-1}$ onto $\mathrm{im}(\chi, \bar{D}(n))$.

$\square$

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