LEMMA 2. *An academic administrative system is compact.*

Proof. The system is governed by a finite number of arbitrarily short-sighted deans and is compact by definition.

Lemmas 1 and 2 verify the hypothesis of Poincaré's recurrence theorem and therefore the conclusions hold for all academic administrations. An immediate consequence of this result is:

THEOREM 1. *Almost all administrators vacillate.*

Finally, since many conservative systems are reversible, an administrator will not only return infinitely often to the same position but must have been there infinitely often in the past.

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SUM-PRESERVING REARRANGEMENTS OF INFINITE SERIES

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1. Introduction. Every student of advanced calculus knows that an absolutely convergent series of real numbers may be rearranged in an arbitrary fashion to obtain a new series which converges to the same sum as that of the original series. The student also finds that conditionally convergent series behave somewhat differently in this regard. Indeed, Riemann proved that such series can be rearranged to converge to any arbitrary real number, or even to diverge. Moreover, it has been shown by J. H. Smith **[8]** that for any conditionally convergent real series and any real number, there is a rearrangement of a prescribed "cycle type" which converges to that number.

Yet, there obviously are rearrangements which *do* preserve the convergence and sum of all infinite series, whether they converge absolutely or merely conditionally. For example, if the series, $u_1+u_2+u_3+\cdots$, converges, then the rearrangement, $u_2+u_3+u_1+u_5+u_6+u_4+\cdots$, is easily seen to converge to the same sum. It would seem reasonable to try to characterize those rearrangements of series which preserve sums of convergent series. This paper surveys the several approaches to the problem to date and gives another characterization of such rearrangements. For the convenience of would-be series rearrangers, five somewhat simpler sufficient but not necessary conditions for "sum-preserving" rearrangements are also developed. The reader is invited to add to this list.

Rearrangements of series can be described in terms of permutations of the positive integers. Let N denote the set of all positive integers. A *permutation* p of N, of course, is a one-to-one mapping of N onto itself. Let p_j be the image of j under the permutation p. The series $\sum u_{p_j}$ is

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In studying questions of rearrangements of series, two essentially different techniques are used. One, which might be called combinatorial, relies heavily upon properties of N and permutations. The other uses results from summability theory about infinite matrix transformations of sequences or series into sequences or series. Both approaches will be illustrated in this paper. Combinatorial methods, although considerably less elegant, are perhaps more suggestive (one can draw pictures!).

2. Levi's Condition. In 1946, F. W. Levi [6] developed the first criterion for a rearrangement to be sum-preserving. Let p be the permutation inducing the rearrangement of a series. For each positive integer n, let s_n be the nth partial sum of the series Σu_j, and let t_n be the nth partial sum of the rearranged series Σu_{p_j}. Observe that the term u_j appears in t_n if and only if $j \in \{p_1, \ldots, p_n\}$, or, equivalently, if and only if $p^{-1}(j) \leqslant n$. Levi calls a term u_j of the original series a "jumping out term" for n if and only if $1 \leqslant j \leqslant n$ and $p^{-1}(j) > n$. These terms appear in s_n but not in t_n. A term u_j is a "jumping in term" for n exactly when $j > n$ and $p^{-1}(j) \leqslant n$, so that such terms appear in t_n but not in s_n. Then, $t_n = s_n + x_n - y_n$, where x_n is the sum of all jumping in terms for n, and y_n is the sum of all jumping out terms for n. A block of consecutive terms appearing in Σu_j is called a "bunch" for n if the block contains only jumping in terms for n or jumping out terms for n, *and* the block is maximal with respect to this property. Let $B(n)$ be the number of bunches for n. With this terminology, Levi's criterion can be stated as follows.

THEOREM 1. *A rearrangement is sum-preserving if and only if for the permutation p inducing the rearrangement there is a positive integer M so that $B(n) \leqslant M$ for each positive integer n.*

Levi's proof shows that when p is any permutation with $B(n) \leqslant M$ for all n, if $s_n \to s$, then $|x_n - y_n| \to 0$, so $t_n \to s$ also. On the other hand, he shows that for any permutation p with $\{B(n)\}$ not uniformly bounded, there is a convergent series whose rearrangement by p does not converge to the sum of the original series.

The proof given by Levi uses combinatorial methods throughout. In 1966, U. C. Guha [4] gave a succinct proof of Levi's theorem using a result of Bosanquet about series-to-sequence (actually, series-to-function) transformations by infinite matrices.

3. Agnew's Condition. Using techniques of summability theory, R. P. Agnew [2] in 1955 found another necessary and sufficient condition for a rearrangement to be sum-preserving. (A similar condition was developed recently by P. A. B. Pleasants [7].)

THEOREM 2. *The rearrangement induced by the permutation p is sum-preserving if and only if there is a positive integer M so that for each positive integer j, the set $\{p_1, \ldots, p_j\}$ is representable as the union of M or fewer blocks of consecutive integers.*

Theorem 1 and Theorem 2 are equivalent, as the following discussion shows. For any permutation p of N and any positive integer n, let O_n be the set of indices j which correspond to jumping-out terms u_j for n, and let I_n be the set of j's corresponding to jumping-in terms u_j for n. The number of bunches for n, $B(n)$, is the number of blocks of consecutive integers whose union is $O_n \cup I_n$. Since $\{p_1, \ldots, p_n\} = ([1, n] - O_n) \cup I_n$, the equivalence of Levi's and Agnew's conditions is readily apparent.

Agnew's proof was based upon regular matrix transformations of sequences into sequences. Later in this paper, a combinatorial proof of Agnew's result is given. It will be seen that this latter proof carries over easily to series in Banach spaces.

It should be pointed out that Pleasants considers rearrangements of series somewhat differently than is done here. In his paper, the permutation p moves the term u_j to where the term u_{p_j} was in the original series. This has the effect of interchanging the roles of the permutation p and its inverse in some of his theorems relative to those given here.

4. Another Condition. One can obtain another characterization of sum-preserving rearrangements in the following way. Let p be the permutation of N which induces the rearrangement. For each $k \in N$, let m_k be the smaller of $p^{-1}(k)$ and $p^{-1}(k+1)$, and let M_k be the larger of these two numbers. Set $I_k = \{j | j \in N \text{ and } m_k \leqslant j < M_k\}$. Note that since p is one-to-one, $I_{k_1} \neq I_{k_2}$ whenever $k_1 \neq k_2$. The criterion can be given in the following way.

THEOREM 3. *The rearrangement induced by the permutation p is sum-preserving if and only if there is a positive integer K so that every collection of K intervals, $\{I_{k_1}, I_{k_2}, \ldots, I_{k_K}\}$, has an empty intersection.*

This result was suggested after consideration of sum-preserving series-to-series matrix transformations. It will be shown later in this paper by combinatorial methods to be equivalent to Theorem 2. A summability proof will also be given so as to illustrate this kind of approach to the problem.

5. Some Elementary Remarks About Permutations. In order to investigate rearrangements more closely, let us make a few observations about permutations of the positive integers.

Let p be a permutation of N and let p_k be the image of the positive integer k under the permutation p. Clearly, $p_k \to \infty$ as $k \to \infty$. Since p is a permutation, it is easy to see that, if n is any positive integer, there is a K so that $n \in \{p_1, \ldots, p_k\}$ for all $k \geqslant K$. Furthermore, for any n we have $\{1, \ldots, n\} \subset \{p_1, \ldots, p_k\}$ for all sufficiently large k.

Any finite subset of N can be represented as the union of a disjoint collection of blocks of consecutive integers. We shall use the notation $J = [c, d]$ for such intervals; so $[c, d] = \{x | x \in N$ and $c \leqslant x \leqslant d\}$, admitting the possibility that $c = d$ as well. If p is any permutation of N, then, corresponding to each $j \in N$, there is a disjoint collection, $\{J(j,1), \ldots, J(j, n_j)\}$ of intervals such that the set $\{p_1, \ldots, p_j\}$ is given by $J(j,1) \cup J(j,2) \cup \cdots \cup J(j, n_j)$. Agnew's criterion in Theorem 2 is that for the permutation p of N the set of integers $\{n_j\}$ is bounded.

Let us arrange the notation so that the interval $J(j, m)$ is to the left of $J(j, n)$ whenever $m < n$. In all of the following discussion, it will be assumed that j is large enough to have $1 \in \{p_1, \ldots, p_j\}$. The interval $J(j,1)$ will always be of the form $[1, b(j)]$ for some integer $b(j)$, where $1 \leqslant b(j) \leqslant j$. The right-hand endpoint of $J(j, n_j)$ will be denoted by $B(j)$, the largest element of the set $\{p_1, \ldots, p_j\}$. Thus, $B(j) \geqslant j$.

6. Sufficient Conditions for the Preservation of Sums. Before getting to the proofs of Theorems 2 and 3, let us give five simple conditions for a rearrangement to preserve sums.

The permutation given in the example mentioned in the Introduction has the property that it does not move any integer very far from its original location in N. In fact, $|p_j - j| \leqslant 2$ for all j. This suggests the following criterion.

CONDITION 1. *There is a positive integer B so that $|p_j - j| \leqslant B$ for all j.*

In fact, as the discussion below shows, a weaker one-sided condition is sufficient for a permutation to produce a sum-preserving rearrangement of a convergent series.

CONDITION 2. *There is a positive integer B so that $p_j \leqslant j + B$ for all j.*

Condition 2 implies that, for each j, the numbers $p_1, \ldots, p_j$ are located somewhere in the interval $[1, j+B]$. We estimate the maximum number of disjoint intervals into which $\{p_1, \ldots, p_j\}$ could be decomposed. If $j + B = 2q$, an even integer, then the worst possible locations of the p_i's (the ones yielding the most intervals) would be to have q of them at $1, 3, 5, \ldots, (j+B-1)$, or at $2, 4, \ldots, (j+B)$. One of the $(j-q)$ remaining p_i's might be located at either end of $[1, j+B]$, but all of the other remaining ones must fall somewhere in the gaps in the first q locations. Thus, the maximum number of intervals in the representation of $\{p_1,\ldots,p_j\}$ when $j+B=2q$, cannot exceed $q-(j-q-1)=B+1$. Similar considerations for $j+B=2q-1$ also yield a bound $B+1$. This shows that, in the notation of the preceding section, $n_j \leqslant B+1$ for all j, so the hypothesis of Theorem 2 is satisfied.

EXAMPLE 1. The permutation p which takes positive integral powers of 2 into their halves and which maps the remaining positive integers onto themselves consecutively satisfies Condition 2 but not Condition 1. We have $p_j=2^{n-1}$ if $j=2^n$, $p_j=2^n+1$ if $j=2^n-1$ and $p_j=j+1$ otherwise. Since $|p_j-j|=2^{n-1}$ when $j=2^n$, Condition 1 fails to hold. But it is easy to see that $p_j \leqslant j+2$ for all j.

Both Levi and Pleasants have observed that the set of all permutations which induce sum-preserving rearrangements of series is not a group. See Pleasants [7] for an example of a sum-preserving rearrangement induced by a permutation whose inverse does not have this property. (Keep in mind that Pleasants's notation differs from that of the other authors mentioned here.) However, Condition 1 is sufficient to assure that both the permutation p and its inverse, p^{-1}, induce sum-preserving rearrangements, for then, both permutations would satisfy Condition 2.

Further discussion of the algebraic structure of the set of convergence-preserving rearrangements of series can be found in [9]. In addition to Condition 1 above, the author mentions two other conditions which are sufficient for both the permutations p and its inverse p^{-1} to be convergence-preserving: (A) $\lim(p_j)/j=1$, and (B) $0<\inf(p_j)/j$ and $\sup(p_j)/j<+\infty$. Clearly, Condition 1 implies (A), which in turn implies (B). See also [10].

It might be suspected that some restriction on the spread of the elements in $\{p_1,\ldots,p_j\}$ would be useful. Such a restriction, in the notation of Section 5 above, might be the following one.

CONDITION 3. *There is a positive integer B so that $B(j)-b(j)\leqslant B$ for all j.*

This condition, in fact, implies Condition 2, since $p_j \leqslant B(j)$ and $b(j) \leqslant j$, but the example given above shows that it is not equivalent to it. For Example 1, when $j=2^n-1$ we have $B(j)-b(j)=(2^n+1)-(2^{n-1}-1)=2^{n-1}+2$.

Condition 3, while being weaker than Condition 2, suggests another approach to the problem at hand. It is possible to give a direct proof that Condition 3 implies that the permutation p yields a sum-preserving rearrangement by observing that for any j,

$$\sum_{k=1}^{j} u_{p_k} = \sum_{k=1}^{b(j)} u_k + \sum{}^{*} u_{p_k},$$

where the second summation on the right is over those p_k's which satisfy $b(j)+1<p_k \leqslant B(j)$. Since $b(j)\to\infty$ as $j\to\infty$, and since $u_k\to 0$ if $\sum u_k$ converges to s, corresponding to any $\varepsilon>0$ one can simultaneously make the first sum on the right differ from s by less than $\varepsilon/2$ in absolute value, and make each term in the second sum less than $\varepsilon/2B$ in absolute value for all $p_k>b(j)+1$ when j is sufficiently large, and complete the proof in the usual way. The key to the success of this approach is *not* the size or growth rate of the $B(j)$'s or p_j's, but rather the cardinality of the set of indices appearing in the second summation, $\sum^*$. This cardinality is, of course, $j-b(j)$.

CONDITION 4. *There is a positive integer C so that $j-b(j)\leqslant C$ for all j.*

Let us also show directly that Condition 4 implies that Agnew's criterion is satisfied. If the intervals $J(j,k)$ in the decomposition of $\{p_1,\ldots,p_j\}$ are written as $J(j,k)=[c_k,d_k]$, $k=2,\ldots,n_j$, their cardinality is (d_k-c_k+1). Since $\{p_1,\ldots,p_j\}$ is the union of (n_j-1) of these disjoint intervals and the interval $[1,b(j)]$, it follows that

$$j=b(j)+\sum_{k=2}^{n_j}(d_k-c_k+1)=b(j)+(n_j-1)+\sum_{k=2}^{n_j}(d_k-c_k)\geqslant b(j)+(n_j-1).$$

If Condition 4 holds, we have $n_j\leqslant C+1$ for all j.

EXAMPLE 2. The permutation which is the inverse of the one given in Example 1 satisfies Condition 4 but not Condition 2. Thus, $p_1=2$, $p_j=2^{n+1}$ when $j=2^n$, $p_j=2^n-1$ if $j=2^n+1$, and $p_j=j-1$ otherwise. Since $p_j=2^n+2^n$ when $j=2^n$, Condition 2 is not met, but it can be seen easily that $j-b(j)\leqslant 2$ for all $j\geqslant 3$.

Not surprisingly, Condition 4 is not necessary for the preservation of sums.

EXAMPLE 3. Let the permutation p be given by the following scheme:

$$N: 1\ 2\ 3\ 4\quad 5\ 6\ 7\ 8\ 9\ 10\ 11\ldots$$

$$p(N): 1\ 4\ 3\ 2\ 10\ 9\ 8\ 7\ 6\quad 5\ 22\ldots$$

Condition 4 does not hold, since $j-b(j)$ is unbounded. (Take $j=3,9,21,\ldots,3\cdot 2^q-3,\ldots$ to see this.) However, it is obvious that $\{p_1,\ldots,p_j\}$ is the union of at most *two* disjoint intervals for each j.

Let us formulate another sufficient condition, this time basing it on Theorem 3. A simple combinatorial argument shows that every sufficiently large positive integer j belongs to exactly $(m+1)(m+2)/2$ different intervals of positive integers whose lengths do not exceed a fixed positive integer m. Thus, for $j>m$, j is in the $(m+1)$ intervals $[j,j],[j,j+1],\ldots,[j,j+m]$, and in the m intervals $[j-1,j],[j-1,j+1],\ldots,[j-1,j-1+m]$, and in the $(m-1)$ intervals $[j-2,j],\ldots,[j-2,j-2+m],\ldots$, and finally in the single interval $[j-m,j]$.

Now, suppose that for the permutation p, all of the intervals I_k defined by p as in Theorem 3 are such that their lengths are bounded by m, say. For any positive integer j, with $j>m$, the collection of all I_k's to which it belongs must be a subset of the set of intervals considered in the preceding paragraph. Hence, any collection of $K=1+(m+1)(m+2)/2$ such intervals must have an empty intersection. These considerations lead to the next condition.

CONDITION 5. *There is a positive integer m so that the lengths of all intervals I_k are bounded by m.*

This, however, is merely a sufficient condition for preservation of sums. For the permutation of Example 3, one can see that the lengths of I_k when $k=4,10,22,46,\ldots$ increase without bound. In fact, the length of I_k when $k=3\cdot 2^q-2$ is $9\cdot 2^{q-1}-2$ for $q\geqslant 1$.

7. The Equivalence of Theorem 2 and Theorem 3. Let us now investigate the relationship between the conditions of Theorem 2 and Theorem 3, in order to show that they are equivalent.

A permutation which is different from the identity mapping obviously cannot preserve the order of N. There are, however, some useful relationships between the ordering of elements of N and the images of elements of N under a permutation p. For example, if q and r are positive integers, then the assertion that $q\leqslant r$ is equivalent to saying that $p_q\in\{p_1,\ldots,p_r\}$, and this is equivalent to the statement $p_r\in\{p_q,p_{q+1},\ldots\}$. Similar assertions hold for strict inequality.

Let p be a permutation of N. Recall that for Theorem 3, $I_k=[m_k,M_k-1]$, where m_k is the smaller of $p^{-1}(k)$ and $p^{-1}(k+1)$, and M_k is the larger of these two numbers. Note that a positive integer j belongs to I_k if and only if either (1) $k\in\{p_1,\ldots,p_j\}$ and $(k+1)\in\{p_{j+1},p_{j+2},\ldots\}$, or (2) $(k+1)\in\{p_1,\ldots,p_j\}$ and $k\in\{p_{j+1},p_{j+2},\ldots\}$.

For any permutation p of N and any positive integer j, we investigate the number of intervals I_k to which j belongs. If the set $\{p_1,\ldots,p_j\}$ is written as the union of the disjoint collection of intervals $[1,b(j)]$, $J(j,2),\ldots,J(j,n_j)$, then we see that $j\in I_k$ in exactly two cases: (a) when

$k \in \{p_1,\ldots,p_j\}$ and k is a right-hand endpoint of one of these intervals, or (b) $(k+1) \in \{p_1,\ldots,p_j\}$ and $(k+1)$ is a left-hand endpoint of some $J(j,i)$. Since the decomposition of the set $\{p_1,\ldots,p_j\}$ into intervals has (n_j-1) gaps, there are exactly $2(n_j-1)+1=2n_j-1$ possible values for k. These considerations show that Theorem 3 is equivalent to Theorem 2.

Theorem 3 requires that any integer j belong to at most $(K-1)$ different intervals I_k, so $2n_j-1 \leqslant K-1$ for all j, and Agnew's criterion in Theorem 2 holds. Obviously, the condition of Theorem 2 implies that of Theorem 3 in view of the preceding discussion.

8. Proof of the Sufficiency of Agnew's Condition. We now prove that a permutation which satisfies the hypothesis of Theorem 2 produces a sum-preserving rearrangement of any convergent series of complex numbers. An examination of the proof shows that it is valid for convergent series in any Banach space (merely use norms instead of absolute values). It should be pointed out that both Levi and Pleasants remark that, because of the combinatorial nature of their proofs, their results hold in more general contexts. Recently, O. Adrian [1] has obtained some sufficient conditions for a permutation to preserve the sum of certain convergent series in Banach spaces, but these results are of a somewhat different character than those discussed here.

Suppose then that p is a permutation of N which satisfies Agnew's condition and that Σu_k is an arbitrary convergent series with sum s. Let $\varepsilon>0$ be given. By the Cauchy Convergence Condition, there is a positive integer m so that for all $n \geqslant m$ and $q \geqslant 1$,

$$\left|\sum_{k=n+1}^{n+q} u_k\right| < \varepsilon/2M,$$

M being the maximum number of intervals into which each $\{p_1,\ldots,p_j\}$ can be decomposed according to Theorem 2. (We refer to the notation of Section 5 above.) If j is large enough so that $b(j) \geqslant m$, then

$$\sum_{k=1}^{j} u_{p_k} - \sum_{k=1}^{b(j)} u_k = \Sigma_2 u_k + \cdots + \Sigma_{n_j} u_k,$$

where Σ_i denotes summation on all k in the interval $J(j,i)$, $i=2,\ldots,n_j$. Each of the sums on the right is less than $\varepsilon/2M$ in absolute value, so it follows that the absolute value of the left-hand side is less than $\varepsilon/2$. Furthermore, it follows from the Cauchy Convergence Condition that for all $n \geqslant m$,

$$\left|\sum_{k=1}^{n} u_k - s\right| \leqslant \varepsilon/2M \leqslant \varepsilon/2.$$

If j is large enough so that $b(j)$ exceeds m, we have

$$\left|\sum_{k=1}^{j} u_{p_k} - s\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so Σu_{p_k} also converges to s.

9. Proof of the Necessity of Agnew's Condition. In order to show that the hypothesis of Theorem 2 is a necessary condition for a rearranged series to converge to the same sum as the original series, we follow some of the ideas in [7]. Before getting to the details, we give some preliminary observations which are valid for any permutation p of N.

Recall from Section 5 that for each j one can write $\{p_1,\ldots,p_j\} = [1, b(j)] \cup J(j,2) \cup \cdots \cup J(j,n_j)$. There are (n_j-1) gaps in this decomposition. Since $B(j)$ is the maximum of $\{p_1,\ldots,p_j\}$ and $B(j) \geqslant j$,

$$\sum_{k=1}^{j} u_k - \sum_{k=1}^{j} u_{p_k} = \left(\sum_{k=1}^{B(j)} u_k - \sum_{k=1}^{j} u_{p_k}\right) + \left(\sum_{k=1}^{j} u_k - \sum_{k=1}^{B(j)} u_k\right) = \Sigma' u_k + \Sigma'' u_k,$$

where the indices of summation in Σ' run over the integers in the gaps of the decomposition of $\{p_1,\ldots,p_j\}$, and $\Sigma'' u_k$ is either zero if $B(j)=j$ or is equal to

$$\left(-\sum_{k=j+1}^{B(j)} u_k\right)$$

otherwise. If Σu_k converges, the Cauchy Convergence Condition assures that $\Sigma'' u_k$ can be made arbitrarily small in absolute value for all sufficiently large j.

Now, suppose that p is a permutation of N which does not satisfy the hypothesis of Theorem 2. We construct a series which converges to zero, but whose rearrangement by the permutation p does not. Specifically, it is shown for a certain convergent series that the absolute value of the summation, $\Sigma' u_k$, is $+1$ for infinitely many values of j. This proof can be modified to hold in a Banach space by selecting a nonzero element x of the space and replacing the terms $+1,-1,+1/2,-1/2,\ldots$ in the construction by $x,-x,\frac{1}{2}x,-\frac{1}{2}x,\ldots$. With this replacement, the summation $\Sigma' u_k$ will have norm equal to the positive number $\|x\|$ for infinitely many j.

Since the condition of Theorem 2 is assumed not to hold for the permutation p, for each positive integer i there are infinitely many positive integers j so that $n_j \geqslant i+2$. Let j_1 be such that $n_{j_1}>3$. Assuming that $j_1,\ldots,j_i$ have been defined, let j_{i+1} be such that $n_{j_{i+1}}>(i+1)+2$ *and* $B(j_i)\leqslant b(j_{i+1})$. This latter condition is possible because $b(j)\to\infty$ as $j\to\infty$. The sequence $\{j_i\}$ of positive integers thus defined is strictly increasing. We construct a convergent series Σu_k as follows.

Let $u_1=1$ and let $u_k=-1$ when $k=b(j_1)+1$. Set $u_k=0$ for all other k satisfying $1\leqslant k\leqslant B(j_1)$. We continue by induction.

If u_k has been defined for all k such that $1\leqslant k\leqslant B(j_{i-1})$, set $u_k=-1/i$ when k is one less than the left-hand endpoint of each of the intervals $J(j_i,q)$, $2\leqslant q\leqslant i+2$; set $u_k=1/i$ when k *is* one of the left-hand endpoints of the same set of intervals; and let u_k equal zero for all other k, $B(j_{i-1})+1\leqslant k\leqslant B(j_i)$. The resulting series, Σu_k, clearly converges to zero, and the $B(j_i)$th partial sums are equal to zero for each i. On the other hand, $\Sigma' u_k=-1$ for $j=j_1,j_2,\ldots$, so it is false that Σu_{p_k} converges to zero.

Although it seems possible, in view of the above proof, that there might be a permutation which preserves the property of convergence of all series but not necessarily convergence to the same sum, it will be shown below that this is not the case. (See also [7].)

10. Summability Considerations. Theorem 2 arose in the context of a matrix transformation of one sequence into another and the consequent requirement that the matrix transforms convergent sequences into convergent sequences. See, for example, [5, p. 43] for these considerations. There is a parallel, almost equivalent theory of matrix transformations of the terms of one series into the terms of another series. In fact, for series with bounded partial sums, the two approaches are equivalent [3, p. 86]. The basic result about "series-to-series" transformations was given by Vermes in [11].

VERMES'S THEOREM. *The infinite matrix (b_{jk}) transforms every convergent series, Σu_k, into a convergent series, Σv_j, where $v_j=\Sigma_k b_{jk}u_k$ if and only if*

(1) *there is an $M>0$ so that $\Sigma_{k=1}^{\infty}|\Sigma_{j=1}^{n}(b_{jk}-b_{j,k+1})|<M$ for every n, and*

(2) *for each k, $\Sigma_{j=1}^{\infty} b_{jk}$ converges to B_k.*

Moreover, Σv_j converges to $B_1 s+\Sigma_{k=1}^{\infty}(B_k-B_{k+1})(s_k-s)$, where s_k is the kth partial sum of Σu_j and s is the sum of the series Σu_j.

Suppose that p is a permutation of N. A rearrangement of the convergent series Σu_k by the permutation p can be considered as a series-to-series transformation in the following way. Set $b_{jk}=1$ if $k=p_j$ and $b_{jk}=0$ if $k\neq p_j$. If $v_j=\Sigma_k b_{jk}u_k$, then $\Sigma v_j=\Sigma u_{p_j}$, the rearrangement produced by p. Since p is one-to-one, each column of (b_{jk}) contains exactly one nonzero term, so $\Sigma_j b_{jk}=1$ for every k, and condition (2) of the theorem is automatically satisfied with $B_k=1$.

The first condition of the theorem can be examined in the following way. Observe that since p is an onto mapping, if k is any arbitrary positive integer, then $k=p_s$ and $(k+1)=p_t$ for some positive integers s and t. If n is any integer, then $\sum_{j=1}^{n} b_{jk}=1$ if $n \geqslant s$ and is 0 if $1 \leqslant n < s$. Similarly, we have $\sum_{j=1}^{n} b_{j,k+1}=1$ if $n \geqslant t$ and is 0 if $1 \leqslant n < t$. Set $f(n,k)=\sum_{j=1}^{n}(b_{jk}-b_{j,k+1})$. Then $f(n,k)=+1$ if $s \leqslant n < t$, $f(n,k)=-1$ if $t \leqslant n < s$, and $f(n,k)=0$ for all other values of n. Hence, in the notation of Theorem 3, $|f(n,k)|=1$ if and only if $n \in I_k$. For each n, the series, $\sum_k |f(n,k)|$, has only finitely many nonzero terms, and its sum is equal to the number of different intervals I_k to which n belongs. The first condition of Vermes's Theorem will be satisfied exactly when there is an upper bound to the number of intervals I_k to which every positive integer n can belong. This requirement is, of course, the hypothesis of Theorem 3.

We observe that the rearranged series, $\sum v_j$, must converge to the same sum as that of $\sum u_k$, since $B_k=1$ for every k.

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MATHEMATICAL NOTES

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ON THE LAW OF LARGE NUMBERS, INFINITE GAMES, AND CATEGORY

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In 1941, J. C. Oxtoby and S. M. Ulam [3, p. 877] showed (via a footnote) that the law of large numbers is false in the sense of category, i.e., the set of real numbers of the unit interval such that in their infinite dyadic development the number of ones in the first n places divided by n tends to one-half is of the first category (although of measure one).